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On clones, minions, and clonoids, especially those of Boolean functions

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Introduction

A **minion** is a set of functions of several arguments from A to B that is closed under taking **minors**, i.e., functions obtained by composing from the right with projections.

For clones C_1 and C_2 on A and B, respectively, a set K of functions from A to B is called a (C_1, C_2)**-clonoid** if $KC_1 \subseteq K$ and $C_2 K \subseteq K$.

The term “minion” was coined and popularized by Opršal around the year 2018.

The term “clonoid” first appeared in a 2016 paper of Aichinger and Mayr.

These concepts have nevertheless appeared in the literature much earlier; see, e.g., Pippenger 2002; Couceiro, Foldes 2009.

They arise naturally in universal algebra, and they have played an important role in the analysis of the computational complexity of CSPs.

Function class composition

Let $f \in \mathcal{F}_{BC}^{(n)}$, $g_1, \dots, g_n \in \mathcal{F}_{AB}^{(m)}$.

The **composition** $f(g_1, \dots, g_n) \in \mathcal{F}_{AC}^{(m)}$ is given by

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})).$$

Let $F \subseteq \mathcal{F}_{BC}$, $G \subseteq \mathcal{F}_{AB}$.

The **composition** FG is defined as

$$FG := \{\, f(g_1, \dots, g_n) \mid n, m \in \mathbb{N}_+,\ f \in F^{(n)},\ g_1, \dots, g_n \in G^{(m)} \,\}.$$

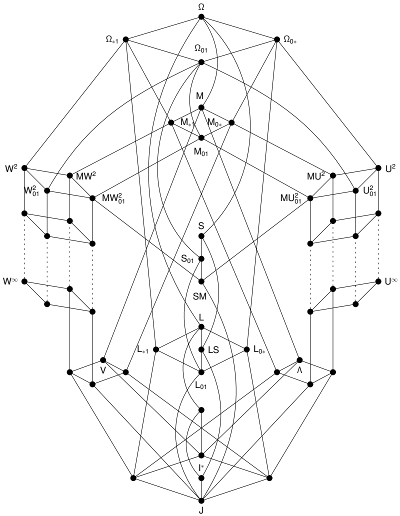
A **clone** is a class $C \subseteq \mathcal{O}_A$ such that $\mathrm{J}_A \subseteq C$ and $CC \subseteq C$.

J_A denotes the class of all projections on A.

For $F \subseteq \mathcal{O}_A$, denote by $\langle F \rangle$ the clone generated by F.

Clones on $\{0,1\}$

Post's lattice



(C_1, C_2)-clonoids

Let $K \subseteq \mathcal{F}_{AB}$.

Let C_1 be a clone on A (the **source clone**), and
let C_2 be a clone on B (the **target clone**).

K is a (C_1, C_2)**-clonoid** (or a (C_1, C_2)**-stable** class)
if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Closure system of (C_1, C_2)-clonoids

For fixed clones C_1 and C_2, the set of all (C_1, C_2)-clonoids constitutes a closure system, i.e.,
– $\mathcal{F}_{AB}$ is a (C_1, C_2)-clonoid,
– arbitrary intersections of (C_1, C_2)-clonoids are (C_1, C_2)-clonoids.

The closure of a set $F \subseteq \mathcal{F}_{AB}$:
$\langle F \rangle_{(C_1,C_2)} = C_2(FC_1)$

Note that $(C_2F)C_1 \subseteq C_2(FC_1)$.
This holds as an equality if F is a minion.
(Couceiro and Foldes's Associativity Lemma)

M. Couceiro, S. Foldes, Functional equations, constraints, definability of function classes, and functions of Boolean variables, *Acta Cybernet.* **18** (2007) 61–75.

Polymorphisms and invariant relations

Let $f \in \mathcal{O}_A^{(n)}$ and let $R \subseteq A^m$.

f **preserves** R (or f is a **polymorphism** of R, or R is an **invariant** of f), in symbols, $f \rhd R$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ ($i \in \{1, \dots, n\}$),

$$f \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{pmatrix} \in R.$$

Galois connection

$$\operatorname{Pol}(\mathcal{R}) := \{\, f \in \mathcal{O}_A \mid \forall R \in \mathcal{R} \colon f \rhd R \,\}$$
$$\operatorname{Inv}(\mathcal{F}) := \{\, R \in \mathcal{R}_A \mid \forall f \in \mathcal{F} \colon f \rhd R \,\}$$

The Galois closed sets of operations are the locally closed clones.

V. G. BODNARČUK, L. A. KALUŽNIN, V. N. KOTOV, B. A. ROMOV, Galois theory for Post algebras, I, II, *Kibernetika* **3** (1969) 1–10, **5** (1969) 1–9 (in Russian); English translation: *Cybernetics* **5** (1969) 243–252, 531–539.

D. GEIGER, Closed systems of functions and predicates, *Pacific J. Math.* **27** (1968) 95–100.

Polymorphisms and invariant relation pairs

Let $f \in \mathcal{F}_{AB}^{(n)}$ and let $R \subseteq A^m$, $S \subseteq B^m$.

f **preserves** (R, S), in symbols, $f \triangleright (R, S)$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ $(i \in \{1, \dots, n\})$,

$$f \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{pmatrix} \in S.$$

Galois connection

$$\mathrm{pPol}(\mathcal{R}) := \{\, f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{R} \colon f \triangleright (R, S) \,\}$$
$$\mathrm{pInv}(\mathcal{F}) := \{\, (R, S) \in \mathcal{R}_{AB} \mid \forall f \in \mathcal{F} \colon f \triangleright (R, S) \,\}$$

The Galois closed sets of functions are the locally closed minions.

N. Pippenger, Galois theory for minors of finite functions, *Discrete Math.* **254** (2002) 405–419.

M. Couceiro, S. Foldes, On closed sets of relational constraints and classes of functions closed under variable substitutions, *Algebra Universalis* **54** (2005) 149–165.

(C_1, C_2)-clonoids and invariant relation pairs

Theorem (Couceiro, Foldes (2009))

Let A and B be arbitrary nonempty sets, and let C_1 and C_2 be clones on A and B, respectively. Let $F \subseteq \mathcal{F}_{AB}$.
The following are equivalent.

1. *F is a locally closed (C_1, C_2)-clonoid.*
2. *F is definable by some set of relation pairs (R, S), where $R \in \operatorname{Inv} C_1$ and $S \in \operatorname{Inv} C_2$.*

M. Couceiro, S. Foldes, Function classes and relational constraints stable under compositions with clones, *Discuss. Math. Gen. Algebra Appl.* **29** (2009) 109–121.

Constraint satisfaction problems

If $\operatorname{Pol}(\mathbf{A}_1) \subseteq \operatorname{Pol}(\mathbf{A}_2)$, then $\operatorname{CSP}(\mathbf{A}_2)$ is reducible to $\operatorname{CSP}(\mathbf{A}_1)$.

If $\operatorname{Pol}(\mathbf{A}_1)$ has a minion homomorphism to $\operatorname{Pol}(\mathbf{A}_2)$, then $\operatorname{CSP}(\mathbf{A}_2)$ is reducible to $\operatorname{CSP}(\mathbf{A}_1)$.

If $\operatorname{pPol}(\mathbf{A}_1, \mathbf{B}_1)$ has a minion homomorphism to $\operatorname{pPol}(\mathbf{A}_2, \mathbf{B}_2)$, then $\operatorname{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ is reducible to $\operatorname{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$.

P. JEAVONS, On the algebraic structure of combinatorial problems, *Theor. Comput. Sci.* **200** (1998) 185–204.

A. BULATOV, P. JEAVONS, A. KROKHIN, Classifying the complexity of constraints using finite algebras, *SIAM J. Comput.* **34** (2005) 720–742.

L. BARTO, J. OPRŠAL, M. PINSKER, The wonderland of reflections, *Israel J. Math.* **223** (2018) 363–298.

J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing* (*STOC '19*), ACM, New York, 2019. pp. 602–613.

L. BARTO, J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *J. ACM* **68** (2021) 1–66.

Question

Question

Given clones C_1 and C_2 on sets A and B, respectively, what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Question

Question

Given clones C_1 and C_2 on finite sets A and B, respectively, what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Question

Question

Given clones C_1 and C_2 on $\{0, 1\}$,
what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Cardinality of the lattice of clonoids

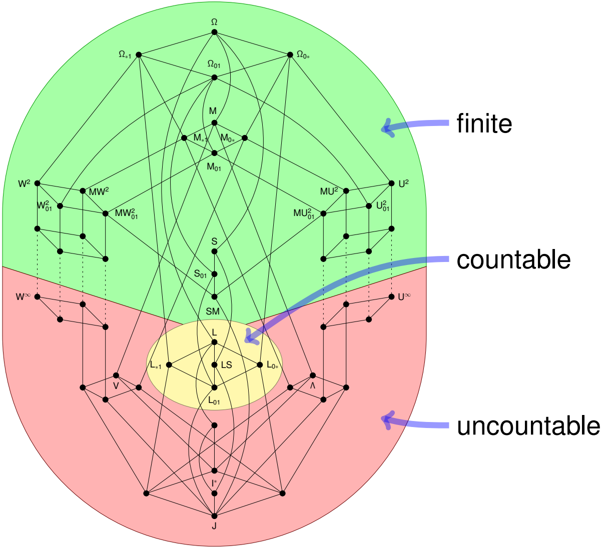
Theorem (Sparks 2019)

Let A be a finite set with $|A| > 1$, and let $B := \{0, 1\}$. Denote by J_A the clone of projections on A, and let C be a clone on B. Then the following statements hold.

1. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *is finite if and only if C contains a near-unanimity operation.*
2. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *is countably infinite if and only if C contains a Mal'cev operation but no majority operation.*
3. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *has the cardinality of the continuum if and only if C contains neither a near-unanimity operation nor a Mal'cev operation.*

A. Sparks, On the number of clonoids, *Algebra Universalis* **80**(4) (2019) Paper No. 53, 10 pp.

Cardinality of the lattice of (J_A, C)-clonoids



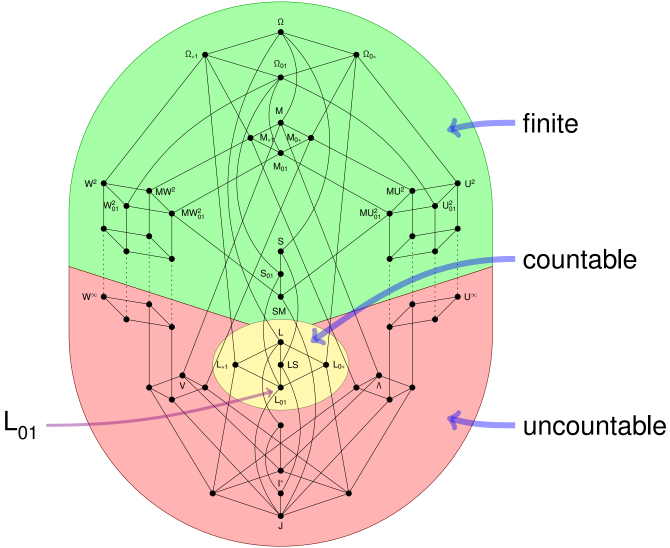
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	?
I	?	?	?	?	?	?	?	?
I^*	?	?	?	?	?	?	?	?
$\Omega(1)$	?	?	?	?	?	?	?	?
V_{01}, Λ_{01}	?	?	?	?	?	?	?	?
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	?
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	?
V, Λ	?	?	?	?	?	?	?	?
MU^k_{01}, MW^k_{01}	?	?	?	?	?	?	?	?
MU^k, MW^k	?	?	?	?	?	?	?	?
U^k_{01}, W^k_{01}	?	?	?	?	?	?	?	?
U^k, W^k	?	?	?	?	?	?	?	?
L_{01}	?	?	?	?	?	?	?	?
L_{0*}, L_{*1}	?	?	?	?	?	?	?	?
LS	?	?	?	?	?	?	?	?
L	?	?	?	?	?	?	?	?
SM	?	?	?	?	?	?	?	?
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	?	?
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	?	?

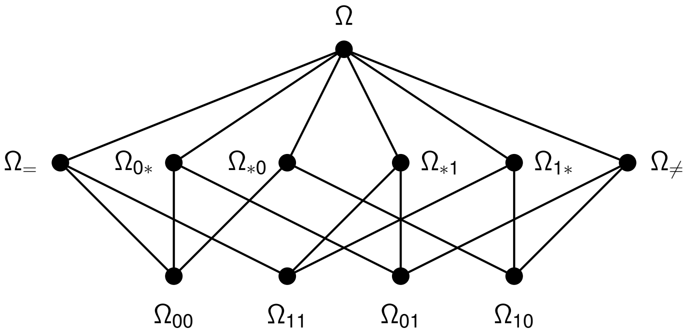
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}^\infty_{01}, \mathrm{MU}^\infty]$ $[\mathrm{MW}^\infty_{01}, \mathrm{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}^k_{01}, \mathrm{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	F
I	?	?	?	?	?	?	?	F
I^*	?	?	?	?	?	?	?	F
$\Omega(1)$	?	?	?	?	?	?	?	F
V_{01}, Λ_{01}	?	?	?	?	?	?	?	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	F
V, Λ	?	?	?	?	?	?	?	F
MU^k_{01}, MW^k_{01}	?	?	?	?	?	?	?	F
MU^k, MW^k	?	?	?	?	?	?	?	F
U^k_{01}, W^k_{01}	?	?	?	?	?	?	?	F
U^k, W^k	?	?	?	?	?	?	?	F
L_{01}	?	?	?	?	?	?	?	F
L_{0*}, L_{*1}	?	?	?	?	?	?	?	F
LS	?	?	?	?	?	?	?	F
L	?	?	?	?	?	?	?	F
SM	?	?	?	?	?	?	?	F
$[\mathrm{M}_{01}, \mathrm{M}]$	?	?	?	?	?	?	?	F
$[\mathrm{S}_{01}, \Omega]$	?	?	?	?	?	?	?	F

Cardinality of the lattice of (J_A, C)-clonoids



$(\mathrm{J}, \mathrm{L}_{01})$-clonoids



$$
\begin{aligned}
\Omega &:= \text{all Boolean functions} \\
\Omega_{a*} &:= \{\, f \in \Omega \mid f(\mathbf{0}) = a \,\} \\
\Omega_{*b} &:= \{\, f \in \Omega \mid f(\mathbf{1}) = b \,\} \\
\Omega_{ab} &:= \Omega_{a*} \cap \Omega_{*b} \\
\Omega_{=} &:= \{\, f \in \Omega \mid f(\mathbf{0}) = f(\mathbf{1}) \,\} \\
\Omega_{\neq} &:= \{\, f \in \Omega \mid f(\mathbf{0}) \neq f(\mathbf{1}) \,\}
\end{aligned}
$$

$(\mathrm{J}, \mathrm{L}_{01})$-clonoids

Zhegalkin polynomial of $f\colon \{0,1\}^n \to \{0,1\}$: $\displaystyle\sum_{S \in M_f} x_S, \quad x_S := \prod_{i \in S} x_i$

degree of f: $\quad \deg(f) := \max_{S \in M_f} |S|$

characteristic rank of f: $\chi(f) := \deg(f + f^{\mathrm{n}}) + 1$

For $i, j \in \mathbb{N}$: $\quad \mathrm{D}_i := \{\, f \in \Omega \mid \deg(f) \leq i \,\} \qquad \mathrm{X}_j := \{\, f \in \Omega \mid \chi(f) \leq j \,\}$

Lemma (Selezneva, Bukhman (2016))

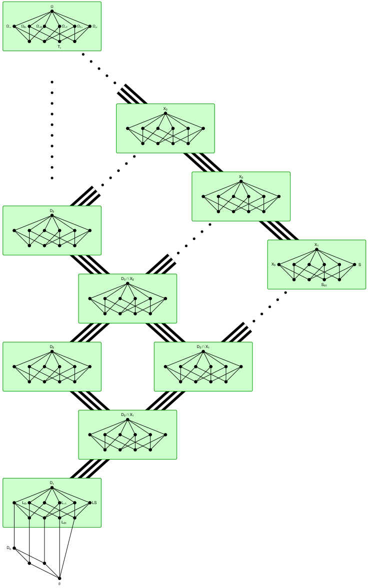
A Boolean function is reflexive if and only if $\chi(f) = 0$.

A Boolean function is self-dual if and only if $f + x_1$ is reflexive.

A Boolean function is self-dual if and only if f is odd and $\chi(f) = 1$.

S. N. Selezneva, A. V. Bukhman, Polynomial-time algorithms for checking some properties of Boolean functions given by polynomials, *Theory Comput. Syst.* **58** (2016) 383–391.

$\aleph_0$ classes



Proof idea

The proof has two parts:

1. Show that these classes are $(\mathrm{J}, \mathrm{L}_{01})$-clonoids.
   This is straightforward verification. It suffices to do this for the meet-irreducible classes.
2. Show that there are no further $(\mathrm{J}, \mathrm{L}_{01})$-clonoids.
   We show that for each class K is generated by any subset of K that is not contained in any lower cover of K (as suggested by the given list of classes).

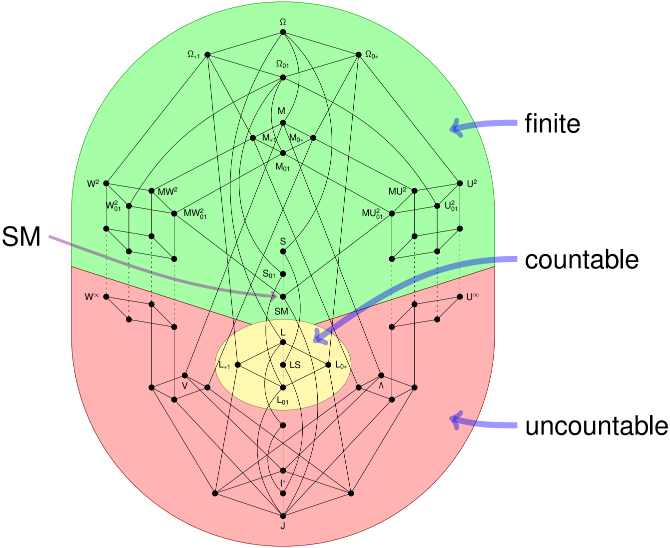
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	F
I	?	?	?	?	?	?	?	F
I^*	?	?	?	?	?	?	?	F
$\Omega(1)$	?	?	?	?	?	?	?	F
V_{01}, Λ_{01}	?	?	?	?	?	?	?	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	F
V, Λ	?	?	?	?	?	?	?	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	?	F
MU^k, MW^k	?	?	?	?	?	?	?	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	?	F
U^k, W^k	?	?	?	?	?	?	?	F
L_{01}	?	?	?	?	?	?	?	F
L_{0*}, L_{*1}	?	?	?	?	?	?	?	F
LS	?	?	?	?	?	?	?	F
L	?	?	?	?	?	?	?	F
SM	?	?	?	?	?	?	?	F
$[\mathrm{M}_{01}, \mathrm{M}]$	?	?	?	?	?	?	?	F
$[\mathrm{S}_{01}, \Omega]$	?	?	?	?	?	?	?	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	C	F
I	?	?	?	?	?	?	C	F
I^*	?	?	?	?	?	?	C	F
$\Omega(1)$	?	?	?	?	?	?	C	F
V_{01}, Λ_{01}	?	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	F	F
V, Λ	?	?	?	?	?	?	F	F
MU^k_{01}, MW^k_{01}	?	?	?	?	?	?	F	F
MU^k, MW^k	?	?	?	?	?	?	F	F
U^k_{01}, W^k_{01}	?	?	?	?	?	?	F	F
U^k, W^k	?	?	?	?	?	?	F	F
L_{01}	?	?	?	?	?	?	C	F
L_{0*}, L_{*1}	?	?	?	?	?	?	C	F
LS	?	?	?	?	?	?	C	F
L	?	?	?	?	?	?	C	F
SM	?	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	F	F

Cardinality of the lattice of (J_A, C)-clonoids



$(\mathsf{J}, \mathsf{SM})$-clonoids

$$\Omega$$

$$\Omega_{\leq} := \{ f \in \Omega \mid f(\mathbf{0}) \leq f(\mathbf{1}) \}$$

$$\Omega_{\geq} := \{ f \in \Omega \mid f(\mathbf{0}) \geq f(\mathbf{1}) \}$$

$$\Omega_{\neq,00} := \Omega_{\neq} \cup \Omega_{00}$$

$$\Omega_{\neq,11} := \Omega_{\neq} \cup \Omega_{11}$$

$$\Omega_{0*} \cup \mathsf{C} \quad \Omega_{*1} \cup \mathsf{C} \quad \Omega_{*0} \cup \mathsf{C} \quad \Omega_{1*} \cup \mathsf{C}$$

$\mathsf{S}^{-} := \{ f \in \Omega \mid \forall \mathbf{a}\colon f(\mathbf{a}) \wedge f(\overline{\mathbf{a}}) = 0 \}$ (minorant of self-dual)

$\mathsf{S}^{+} := \{ f \in \Omega \mid \forall \mathbf{a}\colon f(\mathbf{a}) \vee f(\overline{\mathbf{a}}) = 1 \}$ (majorant of self-dual)

$\mathsf{M} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \leq f(\mathbf{b}) \}$ (monotone)

$\overline{\mathsf{M}} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \geq f(\mathbf{b}) \}$ (antitone)

$\mathsf{U}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon f(\mathbf{a}) = f(\mathbf{b}) = 1 \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$ (1-sep. of rank 2)

$\mathsf{W}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon f(\mathbf{a}) = f(\mathbf{b}) = 0 \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$ (0-sep. of rank 2)

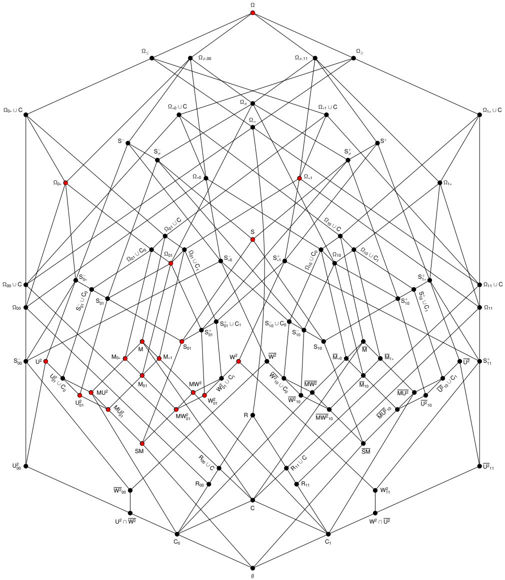
$\overline{\mathsf{U}^2} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon f(\mathbf{a}) = f(\mathbf{b}) = 0 \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$

$\overline{\mathsf{W}^2} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}\colon f(\mathbf{a}) = f(\mathbf{b}) = 1 \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$

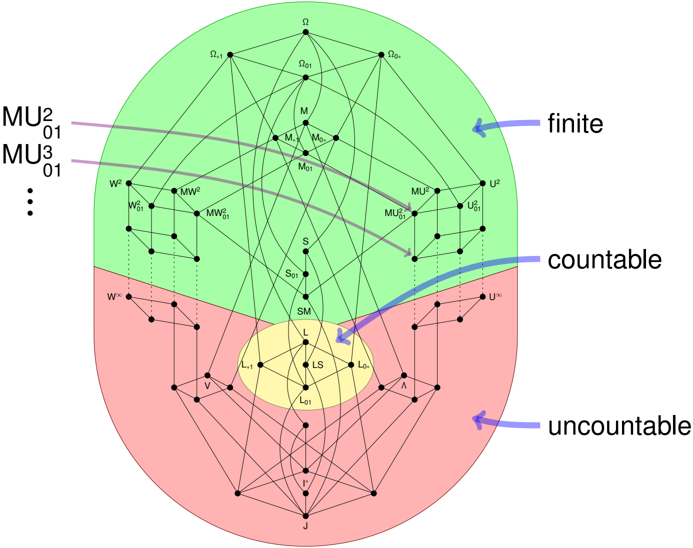
$\mathsf{R} := \{ f \in \Omega \mid \forall \mathbf{a}\colon f(\mathbf{a}) = f(\overline{\mathbf{a}}) \}$ (reflexive)

(J, SM)-clonoids

93 classes

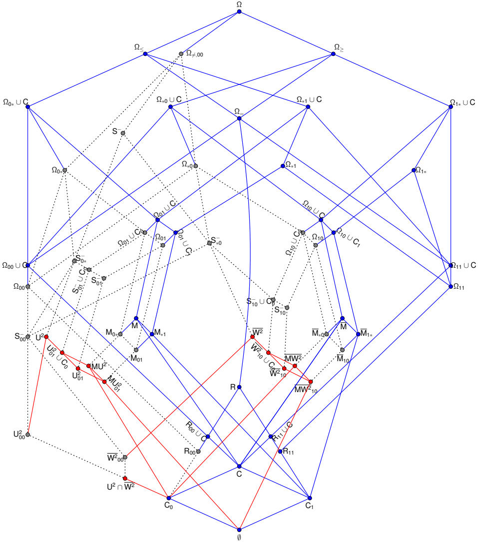


Cardinality of the lattice of (J_A, C)-clonoids



(J, MU^2_{01})-clonoids

63 classes



$(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids

The $(\mathrm{J}, \mathrm{MU}_{01}^{2})$-clonoids are $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids for every $k \geq 2$.

What other $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids are there?

Minorant minions

Definition

Let $f, g\colon \{0,1\}^n \to \{0,1\}$.

f is a **minorant** of g (and g is a **majorant** of f) if $f(\mathbf{a}) \leq g(\mathbf{a})$ for all $\mathbf{a} \in \{0,1\}^n$.

Definition

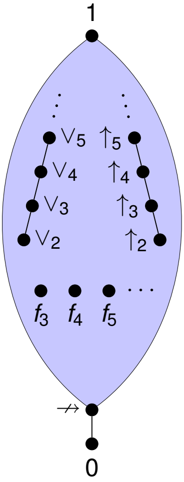
The **minorant-minor** relation $\sqsubseteq$ is the transitive closure of the union of the minorant relation $\leqq$ and the minor relation $\leq$, i.e., $\sqsubseteq = (\leqq \cup \leq)^{\mathrm{tr}}$.

A downset of $(\Omega, \sqsubseteq)$ is called a **minorant minion**.

Proposition

$\sqsubseteq = \leqq \circ \leq$.

Minorant-minor poset



$$\vee_n(a_1, \dots, a_n) = 0 \text{ iff } a_1 = \dots = a_n = 0$$

$$\uparrow_n(a_1, \dots, a_n) = 0 \text{ iff } a_1 = \dots = a_n = 1$$

$$f_n(a_1, \dots, a_n) = 1 \text{ iff } w(\mathbf{a}) \in \{1, n-1\}$$

Schematic Hasse diagram of the minorant-minor poset $(\Omega/{\equiv}, \sqsubseteq)$.

ℓ-local closures

Definition

Let $\Theta \subseteq \Omega$.
For $\ell \in \mathbb{N} \cup \{\infty\}$, the **$\ell$-local closure** of Θ is

$$\ell\text{-Loc}(\Theta) := \{\, f \in \Omega \mid \forall S \subseteq \operatorname{dom}(f)\, (|S| \leq \ell \rightarrow \exists g \in \Theta\, (f|_S = g|_S))\,\}.$$

Proposition

For every $\ell \in \mathbb{N} \cup \{\infty\}$, the mapping ℓ-Loc is a closure operator on Ω.

Proposition

The ℓ-local closure of a minorant minion is a minorant minion.

Notation: $\mathrm{U}_{\Theta}^{\ell} := \ell\text{-Loc}(\downarrow\Theta)$

ℓ-local closures of minorant minions

Proposition

Let $k, \ell \in \mathbb{N} \cup \{\infty\}$ and $\Theta \subseteq \Omega$. If $\max(2, \ell) \leq k$, then the class $\mathsf{U}^{\ell}_{\Theta}$ is a minorant-closed $(\mathsf{J}, \mathsf{MU}^{k}_{01})$-clonoid.

Example

- $\Omega = \mathsf{U}^{1}_{\{1\}}$
- $\Omega_{\neq,00} = \mathsf{U}^{2}_{\{\mathrm{id},+,\neg\}}$
- $\Omega_{0*} = \mathsf{U}^{1}_{\{\mathrm{id}\}}$
- $\Omega_{*0} = \mathsf{U}^{1}_{\{\neg\}}$
- $\Omega_{00} = \mathsf{U}^{1}_{\{\nrightarrow\}} = \mathsf{U}^{2}_{\{+\}}$
- $\mathsf{S}^{-} = \mathsf{U}^{2}_{\{\mathrm{id},\neg\}}$
- $\mathsf{S}^{-}_{0*} = \mathsf{U}^{2}_{\{\mathrm{id},\lambda_{30}\}}$
- $\mathsf{S}^{-}_{*0} = \mathsf{U}^{2}_{\{\neg,\lambda_{31}\}}$
- $\mathsf{S}^{-}_{00} = \mathsf{U}^{2}_{\{\lambda_{30},\lambda_{31}\}}$
- $\mathsf{U}^{\ell} = \mathsf{U}^{\ell}_{\{\mathrm{id}\}}$
- $\overline{\mathsf{W}^{\ell}} = \mathsf{U}^{\ell}_{\{\neg\}}$
- $\mathsf{U}^{\ell} \cap \overline{\mathsf{W}^{\ell}} = \mathsf{U}^{\ell}_{\{\nrightarrow\}}$
- $\mathsf{C}_0 = \mathsf{U}^{1}_{\{0\}}$
- $\emptyset = \mathsf{U}^{1}_{\emptyset}$

ℓ-local closures of minorant minions

Definition

For $\ell \in \mathbb{N} \cup \{\infty\}$, let $\Omega^{[\leq \ell]} := \{\, f \in \Omega \mid |f^{-1}(1)| \leq \ell \,\}$.

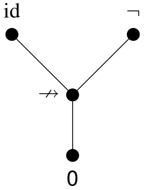
Lemma

Let $k, \ell \in \mathbb{N} \cup \{\infty\}$ with $\max(2, \ell) \leq k$. For every $\Theta \subseteq \Omega$, there exists a $\Psi \subseteq \Omega^{[\leq k]}$ such that $\mathrm{U}^{\ell}_{\Theta} = \mathrm{U}^{k}_{\Psi}$.

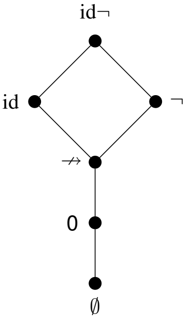
Theorem

Let $k \in \mathbb{N}$. The sets of the form U^{k}_{Θ} for some $\Theta \subseteq \Omega$ constitute a lattice that is isomorphic to $\mathrm{Id}(\Omega^{[\leq k]}/{\equiv}, \sqsubseteq)$.

Minorant-minor poset

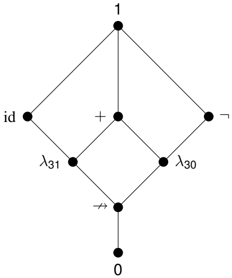


$\mathbf{\Omega}^{[\leq 1]} = (\Omega^{[\leq 1]}, \sqsubseteq)$

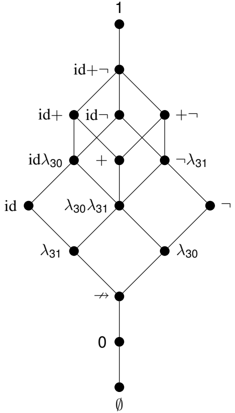


$\mathrm{Id}(\mathbf{\Omega}^{[\leq 1]})$

The minorant-minor poset $\mathbf{\Omega}^{[\leq 1]}$ of Boolean functions with at most one true point and its ideal lattice $\mathrm{Id}(\mathbf{\Omega}^{[\leq 1]})$.



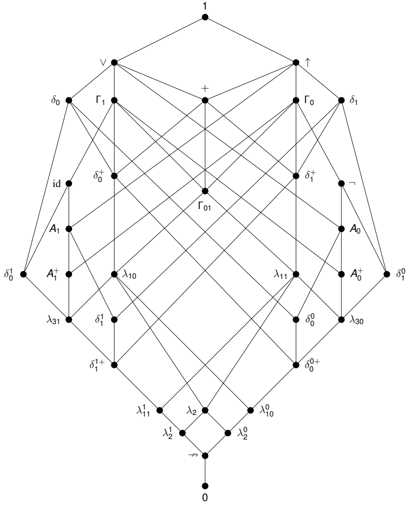
$\mathbf{\Omega}^{[\leq 2]} = (\Omega^{[\leq 2]}, \sqsubseteq)$



$\mathrm{Id}(\mathbf{\Omega}^{[\leq 2]})$

The minorant-minor poset $\mathbf{\Omega}^{[\leq 2]}$ of Boolean functions with at most two true points and its ideal lattice $\mathrm{Id}(\mathbf{\Omega}^{[\leq 2]})$.

Minorant-minor poset



$\mathbf{\Omega}^{[\leq 3]} = (\Omega^{[\leq 3]}, \sqsubseteq)$, the minorant-minor poset of Boolean functions with at most 3 true points. The ideal lattice $\mathrm{Id}(\mathbf{\Omega}^{[\leq 3]})$ has 2854 elements.

$(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids

There are also $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids that are not of the form U_{Θ}^{k}.

$$\begin{aligned}
\Omega &= \text{all Boolean functions} \\
\Omega_{a*} &= \{\, f \mid f(0,\dots,0) = a \,\} & \quad & \Omega_{a*} \cup \mathrm{C} \\
\Omega_{*b} &= \{\, f \mid f(1,\dots,1) = b \,\} & & \Omega_{*b} \cup \mathrm{C} \\
\Omega_{\leq} &= \{\, f \mid f(0,\dots,0) \leq f(1,\dots 1) \,\} \\
\Omega_{\geq} &= \{\, f \mid f(0,\dots,0) \geq f(1,\dots 1) \,\} \\
\mathrm{M} &= \text{monotone functions} \\
\overline{\mathrm{M}} &= \text{anti-monotone functions} \\
\mathrm{R} &= \{\, f \mid \forall \mathbf{a} \in \operatorname{dom}(f)\ f(\mathbf{a}) = f(\overline{\mathbf{a}})\} & & \text{(reflexive functions)}
\end{aligned}$$

Moreover, intersections of $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids are $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids.

$(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids

Definition

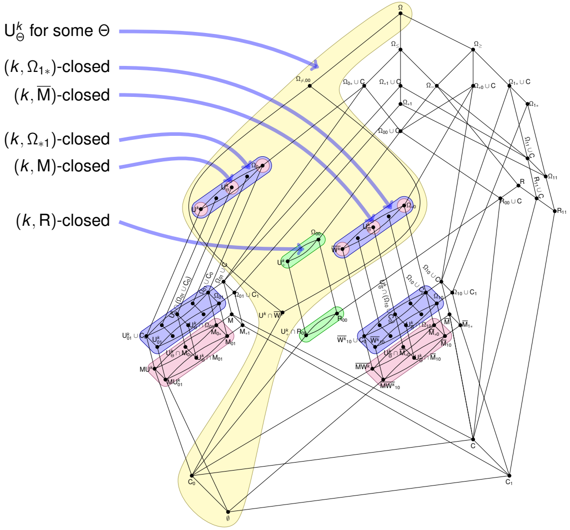
Let $C \in \{\Omega_{*1}, \Omega_{1*}, \mathrm{M}, \overline{\mathrm{M}}, \mathrm{R}\}$. The C**-closure** of f, denoted by f^C, is the least majorant g of f such that $g \in C$.

Definition

Let $\Phi \subseteq \Omega$, and let $C \in \{\Omega_{*1}, \Omega_{1*}, \mathrm{M}, \overline{\mathrm{M}}, \mathrm{R}\}$. The set Φ is (k, C)**-closed** if for every $\varphi \in \Phi$ and for every $T \subseteq (\varphi^C)^{-1}(1)$ with $|T| \leq k$, we have $\chi_T \in \Phi$.

Proposition

Let $k \in \mathbb{N} \cup \{\infty\}$, $C \in \{\Omega_{*1}, \Omega_{1*}, \mathrm{M}, \overline{\mathrm{M}}, \mathrm{R}\}$, and $\Theta \in \mathrm{Id}(\Omega^{[\leq k]})$. Then $\mathrm{U}_{\Theta}^{k} \cap C = (\mathrm{U}_{\Psi}^{k})^C = \mathrm{U}_{\Psi}^{k} \cap C$, where Ψ is the largest (k, C)-closed subset of Θ.



A schematic Hasse diagram of the poset of $(\mathrm{J}, \mathrm{MU}^k_{01})$-clonoids.

$(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids

Theorem

For $k \in \mathbb{N}$ with $k \geq 2$, the $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids are the following:

1. U_{Θ}^{k} *for each* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}^{[\leq k]})$,
2. $\mathrm{U}_{\Theta}^{k} \cap (\Omega_{01} \cup \mathrm{C}_0)$ *and* $\mathrm{U}_{\Theta}^{k} \cap \Omega_{01}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\Omega_{*1}}^{[\leq k]})$,
3. $\mathrm{U}_{\Theta}^{k} \cap (\Omega_{10} \cup \mathrm{C}_0)$ *and* $\mathrm{U}_{\Theta}^{k} \cap \Omega_{10}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\Omega_{1*}}^{[\leq k]})$,
4. $\mathrm{U}_{\Theta}^{k} \cap \mathrm{M}_{0*}$ *and* $\mathrm{U}_{\Theta}^{k} \cap \mathrm{M}_{01}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\mathrm{M}}^{[\leq k]})$,
5. $\mathrm{U}_{\Theta}^{k} \cap \overline{\mathrm{M}}_{*0}$ *and* $\mathrm{U}_{\Theta}^{k} \cap \overline{\mathrm{M}}_{10}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\overline{\mathrm{M}}}^{[\leq k]})$,
6. $\mathrm{U}_{\Theta}^{k} \cap \mathrm{R}_{00}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\mathrm{R}}^{[\leq k]})$,
7. $\Omega_{\leq}$, $\Omega_{\geq}$, $\Omega_{=}$, $\Omega_{0*} \cup \mathrm{C}$, $\Omega_{*0} \cup \mathrm{C}$, $\Omega_{1*} \cup \mathrm{C}$, $\Omega_{*1} \cup \mathrm{C}$, $\Omega_{00} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{11} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}_1$, Ω_{1*}, Ω_{*1}, Ω_{11}, M, M_{*1}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{1*}$, R, $\mathrm{R}_{00} \cup \mathrm{C}$, $\mathrm{R}_{11} \cup \mathrm{C}$, R_{11}, C, C_1.

Sparks's theorem deals with (J, C)-clonoids.

How about (C, J)-clonoids?

C-minors

Let $f\colon A^n \to B$, $g\colon A^m \to B$. Let C be a clone on A. We say that f is a C**-minor** of g, and we write $f \leq_C g$, if $f \in \{g\}C$, i.e., there exist $h_1, \dots, h_m \in C^{(n)}$ such that $f = g(h_1, \dots, h_m)$.

The C-minor relation $\leq_C$ is a quasiorder on $\mathcal{F}_{AB}$.

Note that $\{g\}C = \mathrm{J}_B(\{g\}C) = \langle g \rangle_{(C,\mathrm{J}_B)}$
Therefore, $f \leq_C g$ if and only if $f \in \langle g \rangle_{(C,\mathrm{J}_B)}$.

Consequently, (C, J_B)-clonoids are precisely the downsets of the C-minor quasiorder $\leq_C$.

E. LEHTONEN, Descending chains and antichains of the unary, linear, and monotone subfunction relations, *Order* **23** (2006) 129–142.

E. LEHTONEN, Á. SZENDREI, Equivalence of operations with respect to discriminator clones, *Discrete Math.* **309** (2009) 673–685.

E. LEHTONEN, J. NEŠETŘIL, Minor of Boolean functions with respect to clique functions and hypergraph homomorphisms, *European J. Combin.* **31** (2010) 1981–1995.

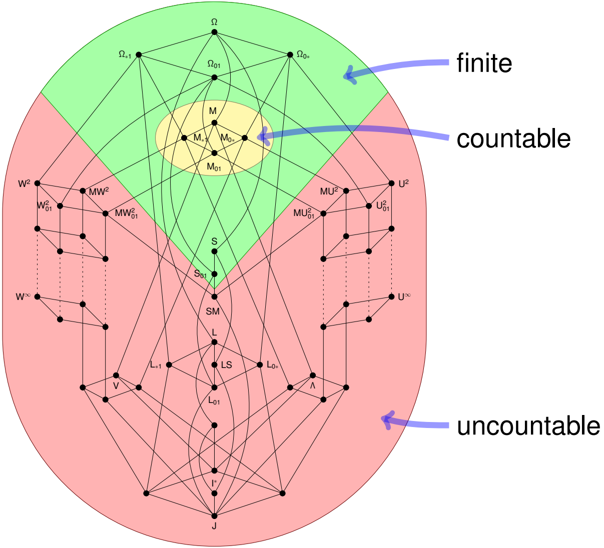
Cardinality of the lattice of (C, J)-clonoids

Theorem

Let C be a clone on $\{0, 1\}$, and let J be the clone of projections on $\{0, 1\}$. Then the following statements hold.

1. $\mathcal{L}_{(C,\mathrm{J})}$ *is finite if and only if C contains the discriminator function.*
2. $\mathcal{L}_{(C,\mathrm{J})}$ *is countably infinite if and only if* $\langle \wedge, \vee \rangle \subseteq C \subseteq \langle \wedge, \vee, 0, 1 \rangle$.
3. $\mathcal{L}_{(C,\mathrm{J})}$ *has the cardinality of the continuum otherwise.*

Cardinality of the lattice of (C, J)-clonoids



Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	C	F
I	?	?	?	?	?	?	C	F
I^*	?	?	?	?	?	?	C	F
$\Omega(1)$	?	?	?	?	?	?	C	F
V_{01}, Λ_{01}	?	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	F	F
V, Λ	?	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	F	F
MU^k, MW^k	?	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	F	F
U^k, W^k	?	?	?	?	?	?	F	F
L_{01}	?	?	?	?	?	?	C	F
L_{0*}, L_{*1}	?	?	?	?	?	?	C	F
LS	?	?	?	?	?	?	C	F
L	?	?	?	?	?	?	C	F
SM	?	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	?	?	?	?	?	?	F	F
$[\mathrm{S}_{01}, \Omega]$	?	?	?	?	?	?	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU^k_{01}, MW^k_{01}	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U^k_{01}, W^k_{01}	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	?	?	?	?	?	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Labeled posets

For $k \in \mathbb{N}$, a ***k*-poset** is a structure $\mathbf{P} = (P, \leq, c)$, where $(P, \leq)$ is a poset and $c\colon P \to [0, k-1]$ is a **labeling.**

Let $\mathbf{P} = (P, \leq, c)$ and $\mathbf{P}' = (P', \leq', c')$ be k-posets. A mapping $h\colon P \to P'$ is a **homomorphism** of $\mathbf{P}$ to $\mathbf{P}'$ if h preserves both the order and the labels, i.e., $h(x) \leq' h(y)$ whenever $x \leq y$, and $c = c' \circ h$.

The homomorphism order of k-posets ($k \geq 2$) has a very rich structure – it is universal for the class of countable posets.

This is the case for k-lattices for $k \geq 3$ as well. However, the homomorphism order of 2-lattices is rather easy to describe.

E. LEHTONEN, Labeled posets are universal, *European J. Combin.* **29** (2008) 493–506.

Clones of monotone functions

Let $(A, \leq)$ be a poset.

Denote by $\bot$ and $\top$ the least and the greatest elements of $(A, \leq)$, provided that such elements exist.

The following sets are clones on A.

M	monotone functions with respect to $\leq$
$M_{\bot}$	monotone $\bot$-preserving functions
$M_{\top}$	monotone $\top$-preserving functions
$M_{\bot\top}$	monotone $\bot$- and $\top$-preserving functions

Monotone minors

Let $(A, \leq)$ be a poset. Assume $A = [0, k-1]$.
We associate with each $f\colon A^n \to A$ the k-poset
$P(f, \leq) := ((A, \leq)^n, f) = (A^n, \leq^n, f)$.

Proposition

Let $(A, \leq)$ be a poset.

1. *$f \leq_M g$ if and only if there exists a homomorphism of $P(f, \leq)$ to $P(g, \leq)$.*
2. *If $(A, \leq)$ has a least element $\bot$, then $f \leq_{M_\bot} g$ if and only if there exists a $\bot$-preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.*
3. *If $(A, \leq)$ has a greatest element $\top$, then $f \leq_{M_\top} g$ if and only if there exists a $\top$-preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.*
4. *If $(A, \leq)$ has both a least element $\bot$ and a greatest element $\top$, then $f \leq_{M_{\bot\top}} g$ if and only if there exists a $\bot$- and $\top$-preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.*

Monotone minors

Proof.

Let C be one of M, $M_\bot$, $M_\top$, $M_{\bot\top}$.

Assume $f \leq_C g$. Then $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in C$. Clearly, $h = (h_1, \dots, h_m)$ is an order-preserving map from $(A, \leq)^n$ to $(A, \leq)^m$ and $f = g \circ h$, so h is a homomorphism of $P(f, \leq)$ to $P(g, \leq)$. Moreover, if $C \subseteq M_\bot$, then $h(\bot, \dots, \bot) = \bot$, so h is $\bot$-preserving. Similarly, if $C \subseteq M_\top$, then $h(\top, \dots, \top) = \top$, so h is $\top$-preserving.

Conversely, assume that there exists a homomorphism h of $P(f, \leq)$ to $P(g, \leq)$. Then $f = g \circ h$, and in $h = (h_1, \dots, h_m)$ each component function h_i is in M. Moreover, if h is $\bot$-preserving, then each h_i is in $M_\bot$; similarly, if h is $\top$-preserving, then each h_i is in $M_\top$. Therefore, $f \leq_C g$. □

Alternating k-chains

A k-chain $(A, \leq, c)$ with $A = \{a_0, a_1, \dots, a_d\}$, $a_0 < a_1 < \cdots < a_d$ is **alternating** if $c(a_i) \neq c(a_{i+1})$ for all $i \in [0, d-1]$.
The number d is the **length** of this chain.

Note that an alternating 2-chain is uniquely determined, up to isomorphism, by its length and the label $c(a_0)$ of its least element; we have $c(a_i) = c(a_0)$ if and only if $i \equiv 0 \pmod 2$.

For $d \in \mathbb{N}$ and $a \in \{0, 1\}$, denote by C_a^d the alternating 2-chain of length d with $c(a_0) = a$.

Alternating k-chains

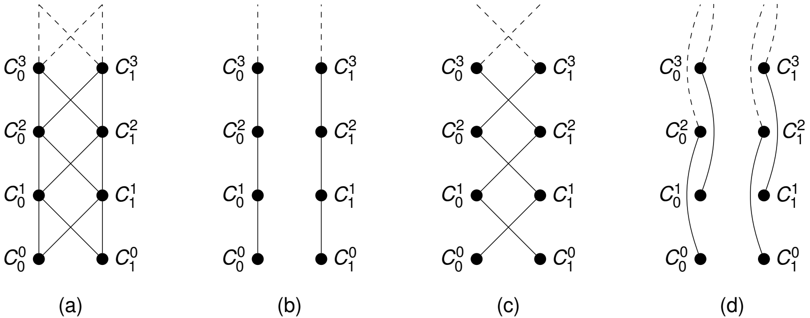
Proposition

Let $d, e \in \mathbb{N}$ and $a, b \in \{0, 1\}$.

1. *There exists a homomorphism of C_a^d to C_b^e if and only if $d < e$ or $(d, a) = (e, b)$.*
2. *There exists a $\bot$-preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$ and $a = b$.*
3. *There exists a $\top$-preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$ and $a + d \equiv b + e \pmod 2$.*
4. *There exists a $\bot$- and $\top$-preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$, $a = b$, and $d \equiv e \pmod 2$.*

Proof.

Straightforward verification. □



Homomorphism orders of 2-chains:
(a) unrestricted,
(b) $\bot$-preserving,
(c) $\top$-preserving, and
(d) $\bot$- and $\top$-preserving homomorphisms.

Homomorphism order of bounded 2-posets

Proposition (Cf. Kosub, Wagner)

Every 2*-poset with a least element* (*or with a greatest element*) *is homomorphically equivalent to its longest alternating* 2*-chain.*

Proof.

Omitted from this presentation. □

S. Kosub, K. W. Wagner, The Boolean hierarchy of NP-partitions, in: H. Reichel, S. Tison (Eds.), STACS 2000, 17th Annual Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Comput. Sci., vol. 1770, Springer-Verlag, Berlin, 2000, pp. 157–168.

S. Kosub, K. W. Wagner, The Boolean hierarchy of NP-partitions, *Inform. and Comput.* **206** (2008) 538–568.

Alternation number of a Boolean function

Let $f\colon \{0,1\}^n \to \{0,1\}$.

$P(k, \leq) := (\{0,1\}^n, \leq^n, f)$

The **alternation number** of f, denoted by $\mathrm{Alt}(f)$, is the length of the longest alternating 2-chain in $P(f, \leq)$, where $\leq$ is the chain $0 < 1$.

Putting together the previous results, we obtain:

Proposition

Let $f, g \in \Omega$, and let $k := \mathrm{Alt}(f)$, $\ell := \mathrm{Alt}(g)$, $a := f(\mathbf{0})$, $b := g(\mathbf{0})$.

1. *$f \leq_{\mathsf{M}} g$ if and only if $k < \ell$ or $(k, a) = (\ell, b)$.*
2. *$f \leq_{\mathsf{M}_{0*}} g$ if and only if $k \leq \ell$ and $a = b$.*
3. *$f \leq_{\mathsf{M}_{*1}} g$ if and only if $k \leq \ell$ and $a + k \equiv b + \ell \pmod 2$.*
4. *$f \leq_{\mathsf{M}_{01}} g$ if and only if $k \leq \ell$, $a = b$, and $k \equiv \ell \pmod 2$.*

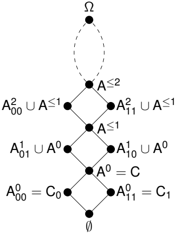
$$\mathsf{A}^k := \{\, f \in \Omega \mid \mathrm{Alt}(f) = k \,\} \qquad \mathsf{A}^{\leq k} := \{\, f \in \Omega \mid \mathrm{Alt}(f) \leq k \,\}$$

(C, J)-clonoids for $C \in \{\mathrm{M}_{01}, \mathrm{M}_{0*}, \mathrm{M}_{*1}, \mathrm{M}\}$

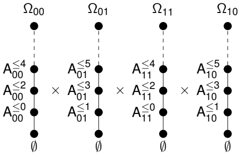
Theorem

1. *The* (M, J)*-clonoids are the sets* $\emptyset$, Ω, $\mathrm{A}_{00}^{0} = \mathrm{C}_0$, $\mathrm{A}_{11}^{0} = \mathrm{C}_1$, $\mathrm{A}^{\leq k}$, $\mathrm{A}_{0*}^{k+1} \cup \mathrm{A}^{\leq k}$, *and* $\mathrm{A}_{1*}^{k+1} \cup \mathrm{A}^{\leq k}$ *for* $k \in \mathbb{N}$.
2. *The* $(\mathrm{M}_{0*}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B$, *where*
$$A \in \{\emptyset, \Omega_{0*}\} \cup \{\, \mathrm{A}_{0*}^{\leq k} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{1*}\} \cup \{\, \mathrm{A}_{1*}^{\leq k} \mid k \in \mathbb{N} \,\}.$$
3. *The* $(\mathrm{M}_{*1}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B$, *where*
$$A \in \{\emptyset, \Omega_{*0}\} \cup \{\, \mathrm{A}_{*0}^{\leq k} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{*1}\} \cup \{\, \mathrm{A}_{*1}^{\leq k} \mid k \in \mathbb{N} \,\}.$$
4. *The* $(\mathrm{M}_{01}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B \cup C \cup D$, *where*
$$A \in \{\emptyset, \Omega_{00}\} \cup \{\, \mathrm{A}_{00}^{\leq k} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{01}\} \cup \{\, \mathrm{A}_{01}^{\leq k} \mid k \in \mathbb{N} \,\},$$
$$C \in \{\emptyset, \Omega_{11}\} \cup \{\, \mathrm{A}_{11}^{\leq k} \mid k \in \mathbb{N} \,\}, \quad D \in \{\emptyset, \Omega_{10}\} \cup \{\, \mathrm{A}_{10}^{\leq k} \mid k \in \mathbb{N} \,\}.$$

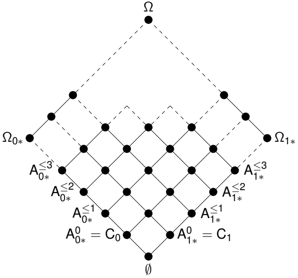
(C, J)-clonoids



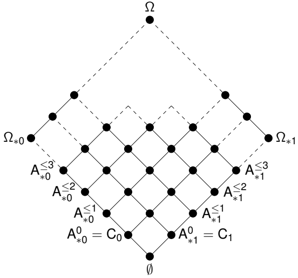
(M, J)-clonoids



(M_{01}, J)-clonoids



(M_{0*}, J)-clonoids



(M_{*1}, J)-clonoids

Proposition

1. *For clones* $C_1 \in \{\mathsf{M}, \mathsf{M}_{0*}, \mathsf{M}_{*1}, \mathsf{M}_{01}\}$*,*
   - *the* (C_1, I_0)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C}_0 = \mathsf{A}^0_{00} \subseteq K$*,*
   - *the* (C_1, I_1)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C}_1 = \mathsf{A}^0_{11} \subseteq K$*,*
   - *the* (C_1, I)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C} = \mathsf{A}^0 \subseteq K$*.*
2. *For clones* $C_1 \in \{\mathsf{M}, \mathsf{M}_{0*}, \mathsf{M}_{*1}\}$ *and* $C_2 \in \{\mathsf{I}^*, \Omega(1)\}$*, the* (C_1, C_2)*-clonoids are the sets* $\emptyset$*,* Ω*,* $\mathsf{A}^{\leq k}$*, for* $k \in \mathbb{N}$*.*
3. *The* $(\mathsf{M}_{01}, \mathsf{I}^*)$*-clonoids are the sets of the form* $A \cup B$*, where*

$$A \in \{\emptyset, \Omega_=\} \cup \{\, \mathsf{A}^{\leq k}_{00} \cup \mathsf{A}^{\leq k}_{11} \mid k \in \mathbb{N} \,\}, \;\; B \in \{\emptyset, \Omega_{\neq}\} \cup \{\, \mathsf{A}^{\leq k}_{01} \cup \mathsf{A}^{\leq k}_{10} \mid k \in \mathbb{N} \,\}.$$

4. *The* $(\mathsf{M}_{01}, \Omega(1))$*-clonoids are* $\emptyset$ *and the sets of the form* $A \cup B$*, where*

$$A \in \{\Omega_=\} \cup \{\, \mathsf{A}^{\leq k}_{00} \cup \mathsf{A}^{\leq k}_{11} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{\neq}\} \cup \{\, \mathsf{A}^{\leq k}_{01} \cup \mathsf{A}^{\leq k}_{10} \mid k \in \mathbb{N} \,\}.$$

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	?	?	?	?	?	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

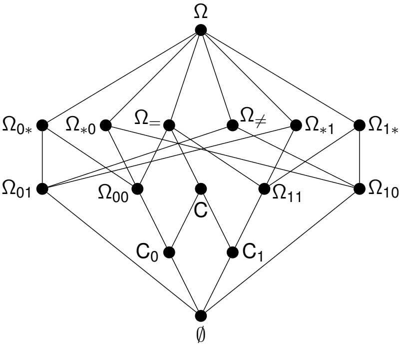
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^{\infty}, \mathsf{MU}^{\infty}]$ $[\mathsf{MW}_{01}^{\infty}, \mathsf{MW}^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^{k}, \mathsf{MW}_{01}^{k}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^{k}, MW_{01}^{k}	U	?	?	?	?	?	F	F
MU^{k}, MW^{k}	U	?	?	?	?	?	F	F
U_{01}^{k}, W_{01}^{k}	U	?	?	?	?	?	F	F
U^{k}, W^{k}	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

$(\mathrm{M}_{01}, \mathrm{L}_{01})$-clonoids

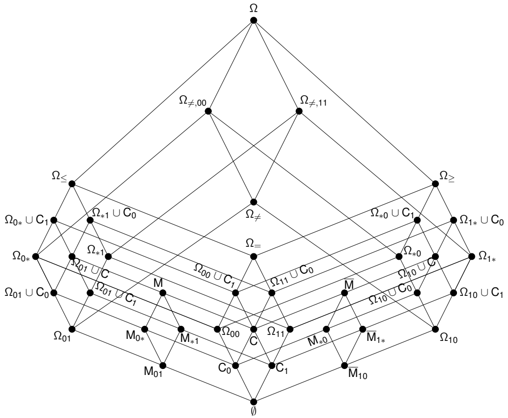
Proposition

There are precisely 15 $(\mathrm{M}_{01}, \mathrm{L}_{01})$*-clonoids, and they are the following:* Ω, Ω_{0*}, Ω_{1*}, Ω_{*0}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, Ω_{00}, Ω_{01}, Ω_{10}, Ω_{11}, C, C_0, C_1, $\emptyset$.



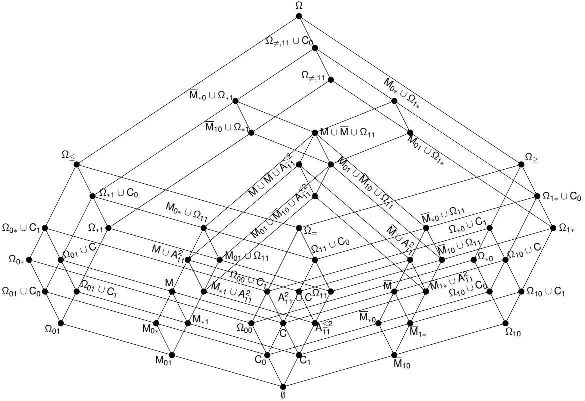
Proposition

There are precisely 39 $(\mathrm{M}_{01}, \mathrm{SM})$*-clonoids, and they are the following:* Ω, $\Omega_{\leq}$, $\Omega_{\geq}$, $\Omega_{\neq,00}$, $\Omega_{\neq,11}$, $\Omega_{0*} \cup \mathrm{C}_1$, $\Omega_{*0} \cup \mathrm{C}_1$, $\Omega_{1*} \cup \mathrm{C}_0$, $\Omega_{*1} \cup \mathrm{C}_0$, Ω_{0*}, Ω_{*0}, Ω_{1*}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_0$, $\Omega_{10} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}_1$, $\Omega_{00} \cup \mathrm{C}_1$, $\Omega_{11} \cup \mathrm{C}_0$, Ω_{00}, Ω_{11}, Ω_{01}, Ω_{10}, M, M_{0*}, M_{*1}, M_{01}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{*0}$, $\overline{\mathrm{M}}_{1*}$, $\overline{\mathrm{M}}_{10}$, C, C_0, C_1, $\emptyset$.



Proposition

There are precisely 56 $(\mathrm{M}_{01}, \mathrm{V}_{01})$-clonoids, and they are the following: Ω, $\Omega_{\neq,11} \cup \mathrm{C}_0$, $\Omega_{\neq,11}$, $\Omega_{\geq}$, $\Omega_{\leq}$, $\Omega_{=}$, $\Omega_{0*} \cup \mathrm{C}_1$, $\Omega_{*0} \cup \mathrm{C}_1$, $\Omega_{1*} \cup \mathrm{C}_0$, $\Omega_{*1} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}_0$, $\Omega_{10} \cup \mathrm{C}_1$, $\Omega_{00} \cup \mathrm{C}_1$, $\Omega_{11} \cup \mathrm{C}_0$, Ω_{0*}, Ω_{*0}, Ω_{1*}, Ω_{*1}, Ω_{01}, Ω_{10}, Ω_{00}, Ω_{11}, $\mathrm{M} \cup \overline{\mathrm{M}} \cup \Omega_{11}$, $\mathrm{M} \cup \overline{\mathrm{M}} \cup \mathrm{A}_{11}^2$, $\mathrm{M}_{01} \cup \overline{\mathrm{M}}_{10} \cup \Omega_{11}$, $\mathrm{M}_{01} \cup \overline{\mathrm{M}}_{10} \cup \mathrm{A}_{11}^{\leq 2}$, $\mathrm{M} \cup \mathrm{A}_{11}^2$, $\mathrm{M}_{0*} \cup \Omega_{1*}$, $\mathrm{M}_{0*} \cup \Omega_{11}$, $\mathrm{M}_{*1} \cup \mathrm{A}_{11}^2$, $\overline{\mathrm{M}} \cup \mathrm{A}_{11}^2$, $\overline{\mathrm{M}}_{*0} \cup \Omega_{*1}$, $\overline{\mathrm{M}}_{*0} \cup \Omega_{11}$, $\overline{\mathrm{M}}_{1*} \cup \mathrm{A}_{11}^2$, $\mathrm{M}_{01} \cup \Omega_{1*}$, $\mathrm{M}_{01} \cup \Omega_{11}$, $\overline{\mathrm{M}}_{10} \cup \Omega_{*1}$, $\overline{\mathrm{M}}_{10} \cup \Omega_{11}$, M, M_{0*}, M_{*1}, M_{01}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{*0}$, $\overline{\mathrm{M}}_{1*}$, $\overline{\mathrm{M}}_{10}$, $\mathrm{A}_{11}^2 \cup \mathrm{C}$, $\mathrm{A}_{11}^{\leq 2}$, C, C_0, C_1, $\emptyset$.



Proposition

There are precisely 56 $(\mathrm{M}_{01}, \Lambda_{01})$-clonoids, and they are the duals of the $(\mathrm{M}_{01}, \mathrm{V}_{01})$-clonoids.

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	F	F	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

A few further finite clonoid lattices

Proposition

There are precisely 7 $(\mathrm{I}, \mathrm{MU}_{01}^{\infty})$*-clonoids, and they are* Ω, M, $\overline{\mathrm{M}}$, C, C_0, C_1*, and* $\emptyset$*. These are also the* $(\mathrm{I}, \mathrm{MW}_{01}^{\infty})$*-clonoids.*

Proposition

There are 13 $(\mathrm{V}_{0*}, \mathrm{MU}_{01}^{\infty})$*-clonoids, and they are* $\emptyset$, C_0, C_1, C, M_{0*}, $\overline{\mathrm{M}}_{1*}$, M, $\overline{\mathrm{M}}$, Ω_{0*}, Ω_{1*}, $\Omega_{0*} \cup \mathrm{C}_1$, $\Omega_{1*} \cup \mathrm{C}_0$, Ω*.*

E. LEHTONEN, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones, arXiv:2412.01107.

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^{\infty}, \mathrm{MU}^{\infty}]$ $[\mathrm{MW}_{01}^{\infty}, \mathrm{MW}^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^{k}, \mathrm{MW}_{01}^{k}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^{k}, MW_{01}^{k}	U	?	?	?	?	?	F	F
MU^{k}, MW^{k}	U	?	?	?	?	?	F	F
U_{01}^{k}, W_{01}^{k}	U	?	?	?	?	?	F	F
U^{k}, W^{k}	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U^k_{01}, W^k_{01}	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	F	F	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Uncountable clonoid lattices

Theorem

For all clones C_1 and C_2 such that $C_1 \subseteq K_1$ and $C_2 \subseteq K_2$ for some $K_1 \in \{\mathsf{U}^2, \mathsf{W}^2\}$ and $K_2 \in \{\mathsf{U}^\infty, \mathsf{W}^\infty\}$, there are an uncountably infinitude of (C_1, C_2)-clonoids.

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}^\infty_{01}, \mathrm{MU}^\infty]$ $[\mathrm{MW}^\infty_{01}, \mathrm{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}^k_{01}, \mathrm{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U^k_{01}, W^k_{01}	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}^\infty_{01}, \mathrm{MU}^\infty]$ $[\mathrm{MW}^\infty_{01}, \mathrm{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}^k_{01}, \mathrm{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	U	U	U	U	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	?	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	U	U	U	U	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	?	U	U	U	U	F	F
MU^k, MW^k	U	?	U	U	U	U	F	F
U^k_{01}, W^k_{01}	U	?	U	U	U	U	F	F
U^k, W^k	U	?	U	U	U	U	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	U	U	U	U	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Uncountable clonoid lattices

Proposition

Let C_1 be a subclone of Ω_{0*} or of Ω_{*1}.

1. $\mathcal{L}_{(C_1,\mathsf{J})}$ *is uncountable if and only if* $\mathcal{L}_{(C_1,\mathsf{I}^*)}$ *is uncountable.*
2. $\mathcal{L}_{(C_1,\mathsf{J})}$ *is countably infinite if and only if* $\mathcal{L}_{(C_1,\mathsf{I}^*)}$ *is countably infinite.*
3. $\mathcal{L}_{(C_1,\mathsf{J})}$ *is finite if and only if* $\mathcal{L}_{(C_1,\mathsf{I}^*)}$ *is finite.*

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ / $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ / $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} / W^∞_{01}	U^∞ / W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	U	U	U	U	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	?	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	U	U	U	U	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	?	U	U	U	U	F	F
MU^k, MW^k	U	?	U	U	U	U	F	F
U^k_{01}, W^k_{01}	U	?	U	U	U	U	F	F
U^k, W^k	U	?	U	U	U	U	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	U	U	U	U	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	F	F	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	U	U	U	U	U	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	U	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	U	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	U	U	U	U	U	F	F
V, Λ	U	?	?	F	F	F	F	F
MU_{01}^k, MW_{01}^k	U	U	U	U	U	U	F	F
MU^k, MW^k	U	U	U	U	U	U	F	F
U_{01}^k, W_{01}^k	U	U	U	U	U	U	F	F
U^k, W^k	U	U	U	U	U	U	F	F
L_{01}	U	U	?	?	?	?	C	F
L_{0*}, L_{*1}	U	U	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Uncountable clonoid lattices

Theorem

For clones C_1 and C_2 such that $C_1 \subseteq K_1$ and $C_2 \subseteq K_2$ for some

$$
\begin{aligned}
(K_1, K_2) \in \{ & (\Omega(1), \Omega(1)), (\Omega(1), \Lambda), (\Omega(1), \mathrm{V}), (\mathrm{I}^*, \mathrm{U}^\infty), (\mathrm{I}^*, \mathrm{W}^\infty), \\
& (\mathrm{I}_0, \mathrm{U}^\infty), (\mathrm{I}_1, \mathrm{U}^\infty), (\mathrm{I}_0, \mathrm{W}^\infty), (\mathrm{I}_1, \mathrm{W}^\infty), (\mathrm{L}, \Omega(1)), \\
& (\Lambda, \Lambda), (\Lambda, \mathrm{V}), (\mathrm{V}, \Lambda), (\mathrm{V}, \mathrm{V}), (\Lambda, \Omega(1)), (\mathrm{V}, \Omega(1)), \\
& (\mathrm{L}, \Lambda), (\mathrm{L}_{0*}, \mathrm{U}^\infty), (\mathrm{L}_{*1}, \mathrm{U}^\infty), (\mathrm{LS}, \mathrm{U}^\infty), \\
& (\mathrm{L}, \mathrm{V}), (\mathrm{L}_{0*}, \mathrm{W}^\infty), (\mathrm{L}_{*1}, \mathrm{W}^\infty), (\mathrm{LS}, \mathrm{W}^\infty) \}
\end{aligned}
$$

there are an uncountable infinitude of (C_1, C_2)-clonoids.

E. LEHTONEN, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones. arXiv:2412.01107

E. LEHTONEN, Clonoids of Boolean functions with a linear source clone and a semilattice or 0- or 1-separating target clone. arXiv:2504.04481

Uncountable clonoid lattices

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1,C_2)} = \langle T \rangle_{(C_1,C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2)-clonoids.

One of the following families of functions does the job in each case.

- $f_n \colon \{0,1\}^n \to \{0,1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n \colon \{0,1\}^n \to \{0,1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n \colon \{0,1\}^n \to \{0,1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

Uncountable clonoid lattices

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1, C_2)} = \langle T \rangle_{(C_1, C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2)-clonoids.

One of the following families of functions does the job in each case.

- $f_n \colon \{0,1\}^n \to \{0,1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n \colon \{0,1\}^n \to \{0,1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n \colon \{0,1\}^n \to \{0,1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}^\infty_{01}, \mathrm{MU}^\infty]$ $[\mathrm{MW}^\infty_{01}, \mathrm{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}^k_{01}, \mathrm{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	U	U	U	U	U	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	U	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	U	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	U	U	U	U	U	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	U	U	U	U	U	F	F
MU^k, MW^k	U	U	U	U	U	U	F	F
U^k_{01}, W^k_{01}	U	U	U	U	U	U	F	F
U^k, W^k	U	U	U	U	U	U	F	F
L_{01}	U	U	?	?	?	?	C	F
L_{0*}, L_{*1}	U	U	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	U	U	U	U	U	C	F
I	U	U	U	F	F	F	C	F
I^*	U	U	U	U	U	U	C	F
$\Omega(1)$	U	U	U	F	F	F	C	F
V_{01}, Λ_{01}	U	U	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	U	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	U	U	U	U	U	F	F
V, Λ	U	U	U	F	F	F	F	F
MU_{01}^k, MW_{01}^k	U	U	U	U	U	U	F	F
MU^k, MW^k	U	U	U	U	U	U	F	F
U_{01}^k, W_{01}^k	U	U	U	U	U	U	F	F
U^k, W^k	U	U	U	U	U	U	F	F
L_{01}	U	U	U	U	U	U	C	F
L_{0*}, L_{*1}	U	U	U	U	U	U	C	F
LS	U	U	U	U	U	U	C	F
L	U	U	U	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

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The End

Thank you for your attention!