

On the cardinalities of clonoid lattices of Boolean functions

Erkko Lehtonen

Department of Mathematics
Khalifa University of Science and Technology
Abu Dhabi, United Arab Emirates

Logic and Discrete Mathematics Seminar

Mathematics Research Centre

Tampere University

20 March 2026

Context

An **operation** on a set A is a mapping $f: A^n \rightarrow A$ for some integer $n \geq 0$, called the **arity** of f .

An **algebraic structure**, or an **algebra**, is a pair $(A, (f_i)_{i \in I})$, where A is a set and $(f_i)_{i \in I}$ is a family of operations of various arities on A .

Example

- A **groupoid** is an algebra (A, \circ) , where \circ is a binary operation.
- A **semigroup** is a groupoid (A, \circ) satisfying the **associative law** $x \circ (y \circ z) = (x \circ y) \circ z$.
- A groupoid is **commutative** if it satisfies the identity $x \circ y = y \circ x$.
- A **semilattice** is a commutative semigroup satisfying the **idempotent law** $x \circ x = x$.
- groups, rings, fields, lattices, vector spaces, modules, Boolean algebras, ...

Universal algebra is a branch of mathematics that studies algebraic structures in a general way, instead of focusing on algebraic structures of a specific type, such as groups or rings.

With the toolbox of universal algebra, we can prove once and for all results that hold for all types of algebraic structures, e.g., the isomorphism theorems.

One of the most remarkable results is Birkhoff's Theorem.

Theorem (Birkhoff, 1935)

A class of algebras (of the same type) is definable by identities if and only if it is closed under homomorphic images, subalgebras, and direct products.

Such classes are called **varieties**.

G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Philos. Soc.* **31**(4) (1935) 434–454.

Clone theory is a subfield of universal algebra that focuses on the study of **clones**, sets of operations on a fixed set that include all projections and are closed under composition.

The set of all term operations of an algebra is a clone. In fact, every clone arises in this way.

It is often seen in universal algebra that many properties of an algebra do not depend so much on its fundamental operations but on its clone of term operations.

Clonoids are a generalization of clones that allows functions from powers of one set (or algebra) into another.

Minions (or **minor-closed classes**) are sets of functions that are closed under formation of **minors**. This is a special case of clonoids.

The terminology we use is quite modern.

- The term “clone” was first used in the 1965 monograph of P. M. Cohn, who attributed it to Philip Hall.
- The term “clonoid” first appeared in a 2016 paper of E. Aichinger and P. Mayr.
- The term “minion” was coined and popularized by J. Opršal around the year 2018.

These concepts have nevertheless appeared in the literature much earlier under different names.

P. M. COHN, *Universal Algebra*, Harper & Row, New York, NY, 1965.

E. AICHINGER, P. MAYR, Finitely generated equational classes, *J. Pure Appl. Algebra* **220** (2016) 2816–2827.

J. BULÍN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, In: *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing (STOC '19)*, June 23–26, 2019, Phoenix, AZ, USA, ACM, New York, NY, USA, pp. 602–613.

A **function (of several arguments)** from A to B is a mapping $f: A^n \rightarrow B$ for some $n \in \mathbb{N}$, called the **arity** of f .

If $A = B$, then we speak of **operations** on A .

Notation: $\mathcal{F}_{AB}^{(n)} := B^{A^n}$, $\mathcal{F}_{AB} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_{AB}^{(n)}$
 $\mathcal{O}_A^{(n)} := \mathcal{F}_{AA}^{(n)}$, $\mathcal{O}_A := \mathcal{F}_{AA}$

The i -th n -ary **projection** on A is the operation $\text{pr}_i^{(n)}: A^n \rightarrow A$, $(a_1, \dots, a_n) \mapsto a_i$.

Denote by J_A the set of all projections on A .

A **function (of several arguments)** from A to B is a mapping $f: A^n \rightarrow B$ for some $n \in \mathbb{N}$, called the **arity** of f .

If $A = B$, then we speak of **operations** on A .

Notation: $\mathcal{F}_{AB}^{(n)} := B^{A^n}$, $\mathcal{F}_{AB} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_{AB}^{(n)}$
 $\mathcal{O}_A^{(n)} := \mathcal{F}_{AA}^{(n)}$, $\mathcal{O}_A := \mathcal{F}_{AA}$

The i -th n -ary **projection** on A is the operation $\text{pr}_i^{(n)}: A^n \rightarrow A$, $(a_1, \dots, a_n) \mapsto a_i$.

Denote by J_A the set of all projections on A .

Composition of functions

Let $f \in \mathcal{F}_{BC}^{(n)}$, $g_1, \dots, g_n \in \mathcal{F}_{AB}^{(m)}$.

The **composition** of f with g_1, \dots, g_n is the function

$f(g_1, \dots, g_n) \in \mathcal{F}_{AC}^{(m)}$ defined by

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})).$$

Example

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f(x_1, x_2, x_3) = x_1 x_2^2 + 4x_3$$

$$g_1: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad g_1(x_1, x_2) = x_1 + 2x_2$$

$$g_2: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad g_2(x_1, x_2) = 3x_1 x_2$$

$$g_3: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad g_3(x_1, x_2) = 5$$

$$\begin{aligned} f(g_1, g_2, g_3)(x_1, x_2) &= (x_1 + 2x_2)(3x_1 x_2)^2 + 4 \cdot 5 \\ &= 9x_1^3 x_2^2 + 18x_1^2 x_2^3 + 20 \end{aligned}$$

Function class composition

Let $F \subseteq \mathcal{F}_{BC}$, $G \subseteq \mathcal{F}_{AB}$.

The **composition** of F with G , denoted by FG , is defined as

$$FG := \{ f(g_1, \dots, g_n) \mid n, m \in \mathbb{N}_+, f \in F^{(n)}, g_1, \dots, g_n \in G^{(m)} \}.$$

Although the composition of functions is associative, function class composition is not. We only have

$$(FG)H \subseteq F(GH).$$

A **clone** is a class $C \subseteq \mathcal{O}_A$ such that $J_A \subseteq C$ and $CC \subseteq C$.

J_A denotes the class of all projections on A .

The set of all clones on A constitutes a closure system, i.e.,

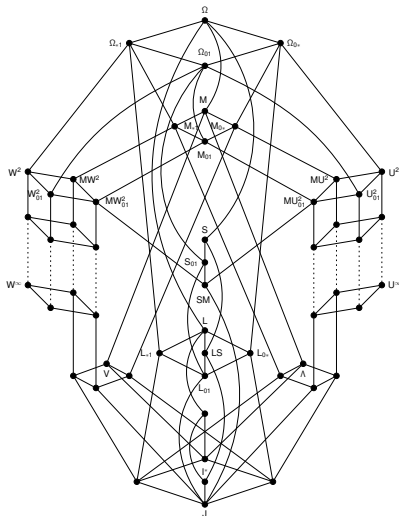
- \mathcal{O}_A is a clone,
- arbitrary intersections of clones are clones.

For $F \subseteq \mathcal{O}_A$, denote by $\langle F \rangle$ the clone generated by F .

Examples of clones:

- \mathcal{O}_A
- \mathbf{J}_A
- polynomial operations on \mathbb{R}
- for a partially ordered set (A, \leq) , monotone (order-preserving) operations
- essentially at most unary operations on A
- idempotent operations on A
- for an algebra $\mathbf{A} = (A, (f_i)_{i \in I})$, the term operations of \mathbf{A}

Post's lattice



E. POST, *The Two-Valued Iterative Systems of Mathematical Logic*, Annals of Mathematical Studies, no. 5, Princeton University Press, Princeton, 1941.

(C_1, C_2) -clonoids

Let $K \subseteq \mathcal{F}_{AB}$.

Let C_1 be a clone on A (the **source clone**), and let C_2 be a clone on B (the **target clone**).

K is a (C_1, C_2) -**clonoid** (or a (C_1, C_2) -**stable class**) if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Special case: (J_A, J_B) -clonoids are called **minions** or **minor-closed classes**.

Let $K \subseteq \mathcal{F}_{AB}$.

Let C_1 be a clone on A (the **source clone**), and let C_2 be a clone on B (the **target clone**).

K is a (C_1, C_2) -**clonoid** (or a (C_1, C_2) -**stable class**) if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Special case: (J_A, J_B) -clonoids are called **minions** or **minor-closed classes**.

Examples of minions

Examples of minions:

- every clone
- polynomial operations on \mathbb{R} of degree at most d
- for a partially ordered set (A, \leq) , order-reversing operations
- threshold functions
- for $a, b \in A$, functions mapping (a, \dots, a) to b
- constant operations
- \emptyset

Closure system of (C_1, C_2) -clonoids

For fixed clones C_1 and C_2 , the set of all (C_1, C_2) -clonoids constitutes a closure system, i.e.,

- \mathcal{F}_{AB} is a (C_1, C_2) -clonoid,
- arbitrary intersections of (C_1, C_2) -clonoids are (C_1, C_2) -clonoids.

The closure of a set $F \subseteq \mathcal{F}_{AB}$:

$$\langle F \rangle_{(C_1, C_2)} = C_2(FC_1)$$

Note that $(C_2F)C_1 \subseteq C_2(FC_1)$.

This holds as an equality if F is a minion.

(Couceiro and Foldes's Associativity Lemma)

M. COUCEIRO, S. FOLDES, Functional equations, constraints, definability of function classes, and functions of Boolean variables, *Acta Cybernet.* **18** (2007) 61–75.

Polymorphisms and invariant relations

Let $f \in \mathcal{O}_A^{(n)}$ and let $R \subseteq A^m$.

f **preserves** R (or f is a **polymorphism** of R , or R is an **invariant** of f), in symbols, $f \triangleright R$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ ($i \in \{1, \dots, n\}$),

$$f \left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{array} \right) := \left(\begin{array}{c} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{array} \right) \in R.$$

Galois connection

$$\text{Pol}(\mathcal{R}) := \{ f \in \mathcal{O}_A \mid \forall R \in \mathcal{R} : f \triangleright R \}$$

$$\text{Inv}(\mathcal{F}) := \{ R \in \mathcal{R}_A \mid \forall f \in \mathcal{F} : f \triangleright R \}$$

The Galois closed sets of operations are the locally closed clones.

V. G. BODNARČUK, L. A. KALUŽNIN, V. N. KOTOV, B. A. ROMOV, Galois theory for Post algebras, I, II, *Kibernetika* **3** (1969) 1–10, **5** (1969) 1–9 (in Russian); English translation: *Cybernetics* **5** (1969) 243–252, 531–539.

D. GEIGER, Closed systems of functions and predicates, *Pacific J. Math.* **27** (1968) 95–100.

Polymorphisms and invariant relation pairs

Let $f \in \mathcal{F}_{AB}^{(n)}$ and let $R \subseteq A^m$, $S \subseteq B^m$.

f **preserves** (R, S) , in symbols, $f \triangleright (R, S)$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ ($i \in \{1, \dots, n\}$),

$$f \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{pmatrix} \in S.$$

Galois connection

$$\text{pPol}(\mathcal{R}) := \{ f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{R}: f \triangleright (R, S) \}$$

$$\text{pInv}(\mathcal{F}) := \{ (R, S) \in \mathcal{R}_{AB} \mid \forall f \in \mathcal{F}: f \triangleright (R, S) \}$$

The Galois closed sets of functions are the locally closed minions.

N. PIPPENGER, Galois theory for minors of finite functions, *Discrete Math.* **254** (2002) 405–419.

M. COUCEIRO, S. FOLDES, On closed sets of relational constraints and classes of functions closed under variable substitutions, *Algebra Universalis* **54** (2005) 149–165.

Theorem (Couceiro, Foldes (2009))

Let A and B be arbitrary nonempty sets, and let C_1 and C_2 be clones on A and B , respectively. Let $F \subseteq \mathcal{F}_{AB}$.

The following are equivalent.

- 1 F is a locally closed (C_1, C_2) -clonoid.
- 2 F is definable by some set of relation pairs (R, S) , where $R \in \text{Inv } C_1$ and $S \in \text{Inv } C_2$.

M. COUCEIRO, S. FOLDES, Function classes and relational constraints stable under compositions with clones, *Discuss. Math. Gen. Algebra Appl.* **29** (2009) 109–121.

Constraint satisfaction problems

Let $\mathbf{A} = (A, (R_i)_{i \in I})$ be a relational structure.

The **constraint satisfaction problem (CSP)** with template \mathbf{A} , $\text{CSP}(\mathbf{A})$, is the following decision problem:

Given a relational structure \mathbf{C} of the same type as \mathbf{A} , decide whether there exists a homomorphism $\mathbf{C} \rightarrow \mathbf{A}$.

The following are natural questions:

- Given a relational structure \mathbf{A} , what is the computational complexity of $\text{CSP}(\mathbf{A})$?
- For which CSP templates \mathbf{A} is $\text{CSP}(\mathbf{A})$ solvable in polynomial time (in P) / nondeterministically in polynomial time (in NP)?
- Given relational structures \mathbf{A} and \mathbf{B} , is $\text{CSP}(\mathbf{A})$ polynomial-time reducible to $\text{CSP}(\mathbf{B})$?

Constraint satisfaction problems

If $\text{Pol}(\mathbf{A}_1) \subseteq \text{Pol}(\mathbf{A}_2)$, then $\text{CSP}(\mathbf{A}_2)$ is reducible to $\text{CSP}(\mathbf{A}_1)$.

If $\text{Pol}(\mathbf{A}_1)$ has a minion homomorphism to $\text{Pol}(\mathbf{A}_2)$, then $\text{CSP}(\mathbf{A}_2)$ is reducible to $\text{CSP}(\mathbf{A}_1)$.

If $\text{pPol}(\mathbf{A}_1, \mathbf{B}_1)$ has a minion homomorphism to $\text{pPol}(\mathbf{A}_2, \mathbf{B}_2)$, then $\text{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ is reducible to $\text{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$.

P. JEAUVONS, On the algebraic structure of combinatorial problems, *Theor. Comput. Sci.* **200** (1998) 185–204.

A. BULATOV, P. JEAUVONS, A. KROKHIN, Classifying the complexity of constraints using finite algebras, *SIAM J. Comput.* **34** (2005) 720–742.

L. BARTO, J. OPRŠAL, M. PINSKER, The wonderland of reflections, *Israel J. Math.* **223** (2018) 363–298.

J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing (STOC '19)*, ACM, New York, 2019. pp. 602–613.

L. BARTO, J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *J. ACM* **68** (2021) 1–66.

Question

Given clones C_1 and C_2 on sets A and B , respectively, what are the (C_1, C_2) -clonoids?

What is the number of (C_1, C_2) -clonoids?

Question

Given clones C_1 and C_2 on finite sets A and B , respectively, what are the (C_1, C_2) -clonoids?

What is the number of (C_1, C_2) -clonoids?

Question

*Given clones C_1 and C_2 on $\{0, 1\}$,
what are the (C_1, C_2) -clonoids?*

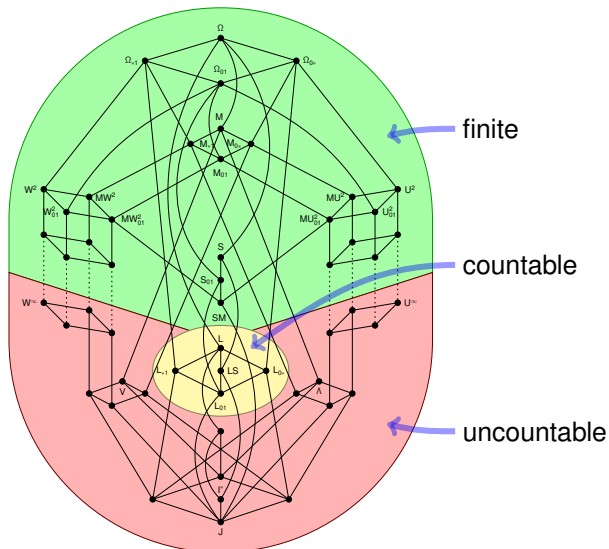
What is the number of (C_1, C_2) -clonoids?

Theorem (Sparks 2019)

Let A be a finite set with $|A| > 1$, and let $B := \{0, 1\}$. Denote by J_A the clone of projections on A , and let C be a clone on B . Then the following statements hold.

- 1 $\mathcal{L}_{(J_A, C)}$ is finite if and only if C contains a near-unanimity operation.
- 2 $\mathcal{L}_{(J_A, C)}$ is countably infinite if and only if C contains a Mal'cev operation but no majority operation.
- 3 $\mathcal{L}_{(J_A, C)}$ has the cardinality of the continuum if and only if C contains neither a near-unanimity operation nor a Mal'cev operation.

Cardinality of the lattice of (J_A, C) -clonoids



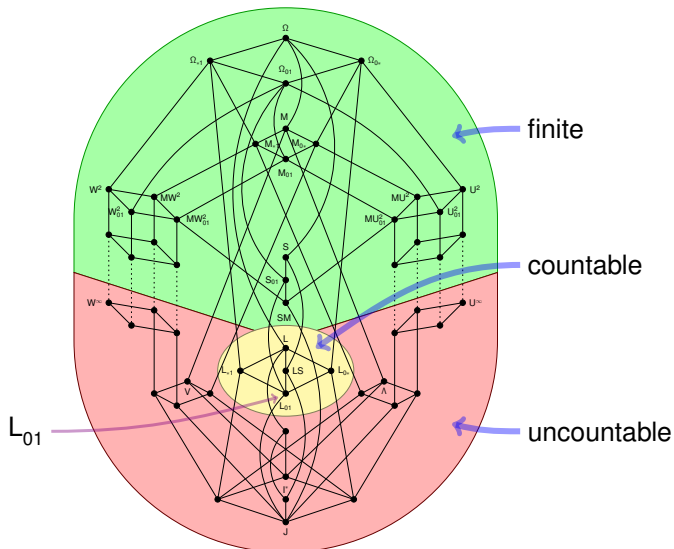
Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | ? | ? | ? | ? | ? | ? | ? | ? |
| I | ? | ? | ? | ? | ? | ? | ? | ? |
| I^* | ? | ? | ? | ? | ? | ? | ? | ? |
| $\Omega(1)$ | ? | ? | ? | ? | ? | ? | ? | ? |
| V_{01}, Λ_{01} | ? | ? | ? | ? | ? | ? | ? | ? |
| V_{0*}, Λ_{*1} | ? | ? | ? | ? | ? | ? | ? | ? |
| V_{*1}, Λ_{0*} | ? | ? | ? | ? | ? | ? | ? | ? |
| V, Λ | ? | ? | ? | ? | ? | ? | ? | ? |
| MU_{01}^k, MW_{01}^k | ? | ? | ? | ? | ? | ? | ? | ? |
| MU^k, MW^k | ? | ? | ? | ? | ? | ? | ? | ? |
| U_{01}^k, W_{01}^k | ? | ? | ? | ? | ? | ? | ? | ? |
| U^k, W^k | ? | ? | ? | ? | ? | ? | ? | ? |
| L_{01} | ? | ? | ? | ? | ? | ? | ? | ? |
| L_{0*}, L_{*1} | ? | ? | ? | ? | ? | ? | ? | ? |
| LS | ? | ? | ? | ? | ? | ? | ? | ? |
| L | ? | ? | ? | ? | ? | ? | ? | ? |
| SM | ? | ? | ? | ? | ? | ? | ? | ? |
| $[M_{01}, M]$ | ? | ? | ? | ? | ? | ? | ? | ? |
| $[S_{01}, \Omega]$ | ? | ? | ? | ? | ? | ? | ? | ? |

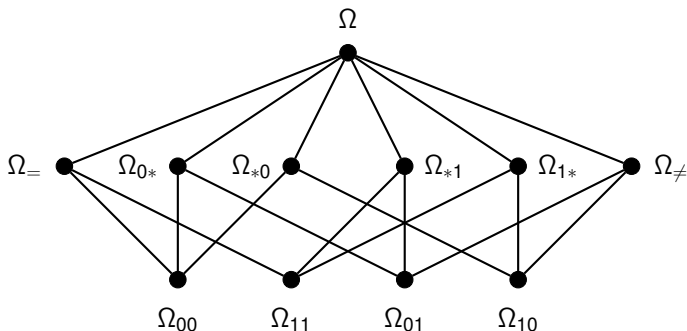
Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | ? | ? | ? | ? | ? | ? | ? | F |
| I | ? | ? | ? | ? | ? | ? | ? | F |
| I^* | ? | ? | ? | ? | ? | ? | ? | F |
| $\Omega(1)$ | ? | ? | ? | ? | ? | ? | ? | F |
| V_{01}, Λ_{01} | ? | ? | ? | ? | ? | ? | ? | F |
| V_{0*}, Λ_{*1} | ? | ? | ? | ? | ? | ? | ? | F |
| V_{*1}, Λ_{0*} | ? | ? | ? | ? | ? | ? | ? | F |
| V, Λ | ? | ? | ? | ? | ? | ? | ? | F |
| MU_{01}^k, MW_{01}^k | ? | ? | ? | ? | ? | ? | ? | F |
| MU^k, MW^k | ? | ? | ? | ? | ? | ? | ? | F |
| U_{01}^k, W_{01}^k | ? | ? | ? | ? | ? | ? | ? | F |
| U^k, W^k | ? | ? | ? | ? | ? | ? | ? | F |
| L_{01} | ? | ? | ? | ? | ? | ? | ? | F |
| L_{0*}, L_{*1} | ? | ? | ? | ? | ? | ? | ? | F |
| LS | ? | ? | ? | ? | ? | ? | ? | F |
| L | ? | ? | ? | ? | ? | ? | ? | F |
| SM | ? | ? | ? | ? | ? | ? | ? | F |
| $[M_{01}, M]$ | ? | ? | ? | ? | ? | ? | ? | F |
| $[S_{01}, \Omega]$ | ? | ? | ? | ? | ? | ? | ? | F |

Cardinality of the lattice of (J_A, C) -clonoids



(J, L_{01}) -clonoids



Ω := all Boolean functions

$\Omega_{a*} := \{ f \in \Omega \mid f(\mathbf{0}) = a \}$

$\Omega_{*b} := \{ f \in \Omega \mid f(\mathbf{1}) = b \}$

$\Omega_{ab} := \Omega_{a*} \cap \Omega_{*b}$

$\Omega_{=} := \{ f \in \Omega \mid f(\mathbf{0}) = f(\mathbf{1}) \}$

$\Omega_{\neq} := \{ f \in \Omega \mid f(\mathbf{0}) \neq f(\mathbf{1}) \}$

(J, L_{01}) -clonoids

Zhegalkin polynomial of $f: \{0, 1\}^n \rightarrow \{0, 1\}$: $\sum_{S \in M_f} x_S$, $x_S := \prod_{i \in S} x_i$

degree of f : $\deg(f) := \max_{S \in M_f} |S|$

characteristic rank of f : $\chi(f) := \deg(f + f^n) + 1$

For $i, j \in \mathbb{N}$: $D_i := \{f \in \Omega \mid \deg(f) \leq i\}$ $X_j := \{f \in \Omega \mid \chi(f) \leq j\}$

Lemma (Selezneva, Bukhman (2016))

A Boolean function is reflexive if and only if $\chi(f) = 0$.

A Boolean function is self-dual if and only if $f + x_1$ is reflexive.

A Boolean function is self-dual if and only if f is odd and $\chi(f) = 1$.

S. N. SELEZNEVA, A. V. BUKHMAN, Polynomial-time algorithms for checking some properties of Boolean functions given by polynomials, *Theory Comput. Syst.* **58** (2016) 383–391.

Proving that $\mathcal{L}_{(C_1, C_2)}$ is finite or countably infinite

A typical proof of a result describing $\mathcal{L}_{(C_1, C_2)}$ has three parts:

- 1 *Find the (C_1, C_2) -clonoids.*

This requires some creativity and iterations when the 3rd step doesn't seem to work.

- 2 *Show that these classes are indeed (C_1, C_2) -clonoids.*

This is straightforward verification. It suffices to do this for the meet-irreducible classes.

- 3 *Show that there are no further (C_1, C_2) -clonoids.*

This is hard work. We show that for each class K is generated by any subset of K that is not contained in any lower cover of K (as suggested by the given list of classes).

Applying monotonicity of function class composition

Let C_1, C'_1, C_2, C'_2 be clones such that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$.

It follows from the monotonicity of function class composition that every (C'_1, C'_2) -clonoid is a (C_1, C_2) -clonoid.

(Proof. Assume K is a (C'_1, C'_2) -clonoid. Then $KC_1 \subseteq KC'_1 \subseteq K$ and $C_2K \subseteq C'_2K \subseteq K$, so K is a (C_1, C_2) -clonoid. \square)

Assume the (C_1, C_2) -clonoids are known.

In order to determine the (C'_1, C'_2) -clonoids, it suffices to identify them among the (C_1, C_2) -clonoids.

Applying monotonicity of function class composition

Let C_1, C'_1, C_2, C'_2 be clones such that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$.

It follows from the monotonicity of function class composition that every (C'_1, C'_2) -clonoid is a (C_1, C_2) -clonoid.

(Proof. Assume K is a (C'_1, C'_2) -clonoid. Then $KC_1 \subseteq KC'_1 \subseteq K$ and $C_2K \subseteq C'_2K \subseteq K$, so K is a (C_1, C_2) -clonoid. \square)

Assume the (C_1, C_2) -clonoids are known.

In order to determine the (C'_1, C'_2) -clonoids, it suffices to identify them among the (C_1, C_2) -clonoids.

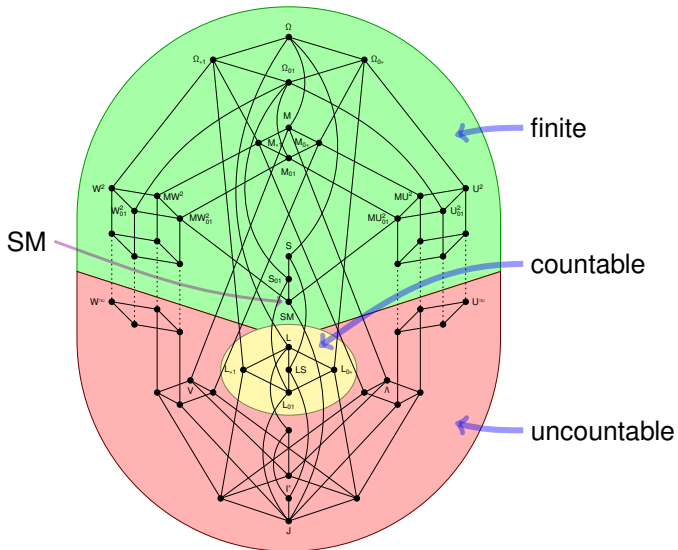
Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | ? | ? | ? | ? | ? | ? | ? | F |
| I | ? | ? | ? | ? | ? | ? | ? | F |
| I^* | ? | ? | ? | ? | ? | ? | ? | F |
| $\Omega(1)$ | ? | ? | ? | ? | ? | ? | ? | F |
| V_{01}, Λ_{01} | ? | ? | ? | ? | ? | ? | ? | F |
| V_{0*}, Λ_{*1} | ? | ? | ? | ? | ? | ? | ? | F |
| V_{*1}, Λ_{0*} | ? | ? | ? | ? | ? | ? | ? | F |
| V, Λ | ? | ? | ? | ? | ? | ? | ? | F |
| MU_{01}^k, MW_{01}^k | ? | ? | ? | ? | ? | ? | ? | F |
| MU^k, MW^k | ? | ? | ? | ? | ? | ? | ? | F |
| U_{01}^k, W_{01}^k | ? | ? | ? | ? | ? | ? | ? | F |
| U^k, W^k | ? | ? | ? | ? | ? | ? | ? | F |
| L_{01} | ? | ? | ? | ? | ? | ? | ? | F |
| L_{0*}, L_{*1} | ? | ? | ? | ? | ? | ? | ? | F |
| LS | ? | ? | ? | ? | ? | ? | ? | F |
| L | ? | ? | ? | ? | ? | ? | ? | F |
| SM | ? | ? | ? | ? | ? | ? | ? | F |
| $[M_{01}, M]$ | ? | ? | ? | ? | ? | ? | ? | F |
| $[S_{01}, \Omega]$ | ? | ? | ? | ? | ? | ? | ? | F |

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | ? | ? | ? | ? | ? | ? | C | F |
| I | ? | ? | ? | ? | ? | ? | C | F |
| I^* | ? | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | ? | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | ? | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | ? | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | ? | ? | ? | ? | ? | ? | F | F |
| V, Λ | ? | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | ? | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | ? | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | ? | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | ? | ? | ? | ? | ? | ? | F | F |
| L_{01} | ? | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | ? | ? | ? | ? | ? | ? | C | F |
| LS | ? | ? | ? | ? | ? | ? | C | F |
| L | ? | ? | ? | ? | ? | ? | C | F |
| SM | ? | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | ? | ? | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | ? | ? | ? | ? | ? | ? | F | F |

(J, SM)-clonoids



(J, SM)-clonoids

 Ω

$$\Omega_{\leq} := \{ f \in \Omega \mid f(\mathbf{0}) \leq f(\mathbf{1}) \}$$

$$\Omega_{\geq} := \{ f \in \Omega \mid f(\mathbf{0}) \geq f(\mathbf{1}) \}$$

$$\Omega_{\neq,00} := \Omega_{\neq} \cup \Omega_{00}$$

$$\Omega_{\neq,11} := \Omega_{\neq} \cup \Omega_{11}$$

$$\Omega_{0*} \cup \mathbf{C} \quad \Omega_{*1} \cup \mathbf{C} \quad \Omega_{*0} \cup \mathbf{C} \quad \Omega_{1*} \cup \mathbf{C}$$

$$\mathbf{S}^- := \{ f \in \Omega \mid \forall \mathbf{a}: f(\mathbf{a}) \wedge f(\bar{\mathbf{a}}) = \mathbf{0} \}$$

(minorant of self-dual)

$$\mathbf{S}^+ := \{ f \in \Omega \mid \forall \mathbf{a}: f(\mathbf{a}) \vee f(\bar{\mathbf{a}}) = \mathbf{1} \}$$

(majorant of self-dual)

$$\mathbf{M} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \leq f(\mathbf{b}) \}$$

(monotone)

$$\bar{\mathbf{M}} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \geq f(\mathbf{b}) \}$$

(antitone)

$$\mathbf{U}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: f(\mathbf{a}) = f(\mathbf{b}) = \mathbf{1} \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$$

(1-sep. of rank 2)

$$\mathbf{W}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: f(\mathbf{a}) = f(\mathbf{b}) = \mathbf{0} \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$$

(0-sep. of rank 2)

$$\bar{\mathbf{U}}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: f(\mathbf{a}) = f(\mathbf{b}) = \mathbf{0} \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$$

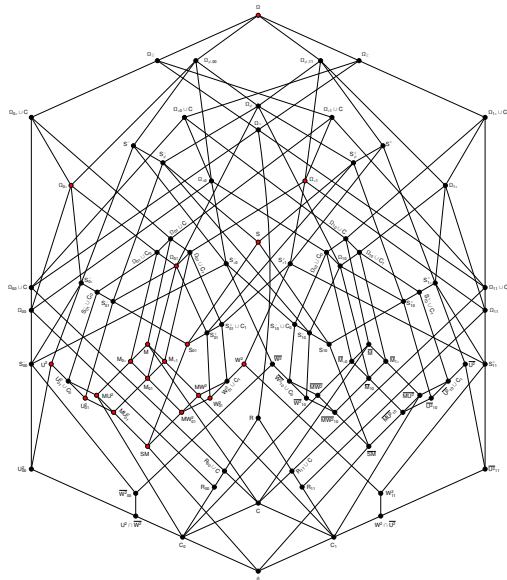
$$\bar{\mathbf{W}}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b}: f(\mathbf{a}) = f(\mathbf{b}) = \mathbf{1} \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$$

$$\mathbf{R} := \{ f \in \Omega \mid \forall \mathbf{a}: f(\mathbf{a}) = f(\bar{\mathbf{a}}) \}$$

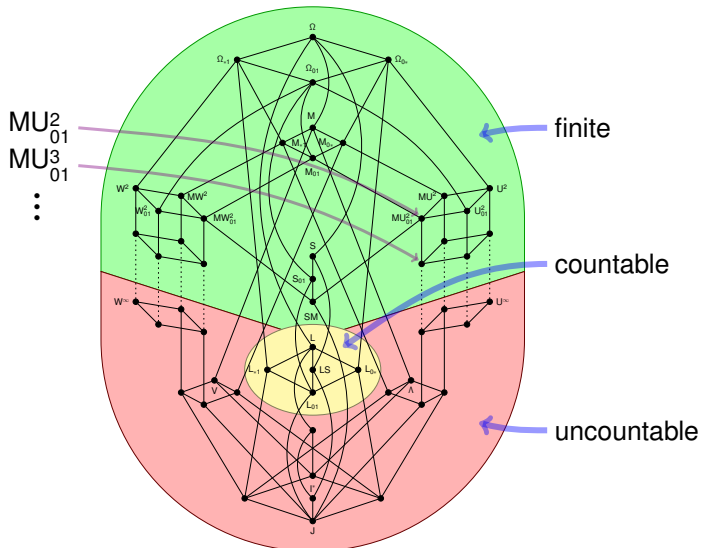
(reflexive)

(J, SM)-clonoids

93 classes

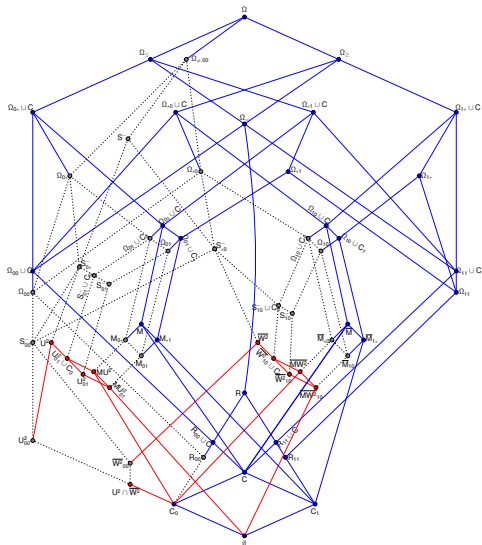


(J, MU_{01}^k) -clonoids, $k \geq 2$



(J, MU_{01}^2) -clonoids

63 classes



The (J, MU_{01}^2) -clonoids are (J, MU_{01}^k) -clonoids for every $k \geq 2$.

What other (J, MU_{01}^k) -clonoids are there?

Minorant minions

Definition

Let $f, g: \{0, 1\}^n \rightarrow \{0, 1\}$.

f is a **minorant** of g (and g is a **majorant** of f)

if $f(\mathbf{a}) \leq g(\mathbf{a})$ for all $\mathbf{a} \in \{0, 1\}^n$.

Definition

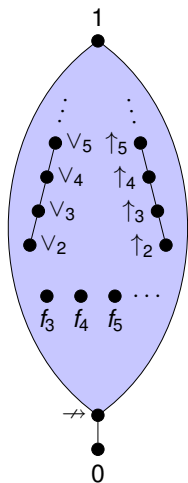
The **minorant-minor** relation \sqsubseteq is the transitive closure of the union of the minorant relation \leq and the minor relation \leq , i.e., $\sqsubseteq = (\leq \cup \leq)^{\text{tr}}$.

A downset of (Ω, \sqsubseteq) is called a **minorant minion**.

Proposition

$$\sqsubseteq = \leq \circ \leq.$$

Minorant-minor poset



$$\vee_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = 0 \text{ iff } \mathbf{a}_1 = \dots = \mathbf{a}_n = 0$$

$$\uparrow_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = 0 \text{ iff } \mathbf{a}_1 = \dots = \mathbf{a}_n = 1$$

$$f_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = 1 \text{ iff } w(\mathbf{a}) \in \{1, n-1\}$$

Schematic Hasse diagram of the minorant-minor poset $(\Omega/\equiv, \sqsubseteq)$.

Definition

Let $\Theta \subseteq \Omega$.

For $l \in \mathbb{N} \cup \{\infty\}$, the **l -local closure** of Θ is

$$l\text{-Loc}(\Theta) := \{f \in \Omega \mid \forall S \subseteq \text{dom}(f) (|S| \leq l \rightarrow \exists g \in \Theta (f|_S = g|_S))\}.$$

Proposition

For every $l \in \mathbb{N} \cup \{\infty\}$, the mapping $l\text{-Loc}$ is a closure operator on Ω .

Proposition

The l -local closure of a minorant minion is a minorant minion.

Notation: $U_{\Theta}^l := l\text{-Loc}(\downarrow\Theta)$

ℓ -local closures of minorant minions

Proposition

Let $k, \ell \in \mathbb{N} \cup \{\infty\}$ and $\Theta \subseteq \Omega$. If $\max(2, \ell) \leq k$, then the class U_{Θ}^{ℓ} is a minorant-closed $(\mathbf{J}, \text{MU}_{01}^k)$ -clonoid.

Example

- $\Omega = U_{\{1\}}^1$
- $\Omega_{\neq, 00} = U_{\{\text{id}, +, \neg\}}^2$
- $\Omega_{0*} = U_{\{\text{id}\}}^1$
- $\Omega_{*0} = U_{\{\neg\}}^1$
- $\Omega_{00} = U_{\{\leftrightarrow\}}^1 = U_{\{+\}}^2$
- $S^- = U_{\{\text{id}, \neg\}}^2$
- $S_{0*}^- = U_{\{\text{id}, \lambda_{30}\}}^2$
- $S_{*0}^- = U_{\{\neg, \lambda_{31}\}}^2$
- $S_{00}^- = U_{\{\lambda_{30}, \lambda_{31}\}}^2$
- $U^{\ell} = U_{\{\text{id}\}}^{\ell}$
- $\overline{W}^{\ell} = U_{\{\neg\}}^{\ell}$
- $U^{\ell} \cap \overline{W}^{\ell} = U_{\{\leftrightarrow\}}^{\ell}$
- $C_0 = U_{\{0\}}^1$
- $\emptyset = U_{\emptyset}^1$

Definition

For $l \in \mathbb{N} \cup \{\infty\}$, let $\Omega^{[\leq l]} := \{f \in \Omega \mid |f^{-1}(1)| \leq l\}$.

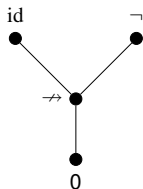
Lemma

Let $k, l \in \mathbb{N} \cup \{\infty\}$ with $\max(2, l) \leq k$. For every $\Theta \subseteq \Omega$, there exists a $\Psi \subseteq \Omega^{[\leq k]}$ such that $U_{\Theta}^l = U_{\Psi}^k$.

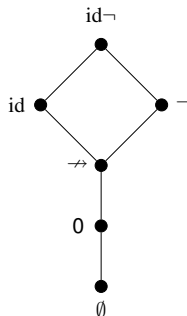
Theorem

Let $k \in \mathbb{N}$. The sets of the form U_{Θ}^k for some $\Theta \subseteq \Omega$ constitute a lattice that is isomorphic to $\text{Id}(\Omega^{[\leq k]}) / \equiv, \sqsubseteq$.

Minorant-minor poset



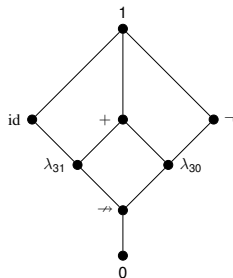
$$\Omega^{[\leq 1]} = (\Omega^{[\leq 1]}, \sqsubseteq)$$



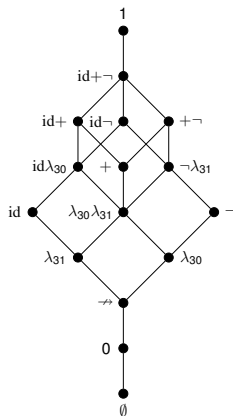
$$\text{Id}(\Omega^{[\leq 1]})$$

The minorant-minor poset $\Omega^{[\leq 1]}$ of Boolean functions with at most one true point and its ideal lattice $\text{Id}(\Omega^{[\leq 1]})$.

Minorant-minor poset



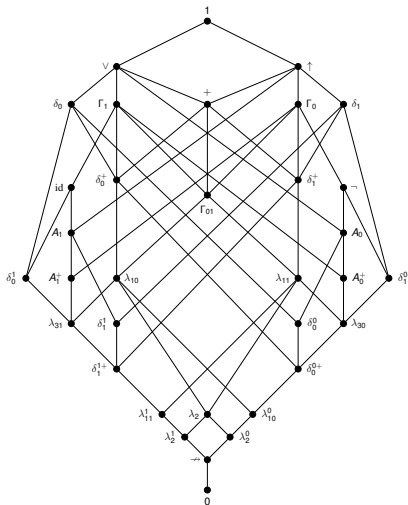
$$\Omega^{[\leq 2]} = (\Omega^{[\leq 2]}, \subseteq)$$



$$\text{Id}(\Omega^{[\leq 2]})$$

The minorant-minor poset $\Omega^{[\leq 2]}$ of Boolean functions with at most two true points and its ideal lattice $\text{Id}(\Omega^{[\leq 2]})$.

Minorant-minor poset



$\Omega^{[\leq 3]} = (\Omega^{[\leq 3]}, \sqsubseteq)$, the minorant-minor poset of Boolean functions with at most 3 true points. The ideal lattice $\text{Id}(\Omega^{[\leq 3]})$ has 2854 elements.

(J, MU_{01}^k) -clonoids

There are also (J, MU_{01}^k) -clonoids that are not of the form U_{Θ}^k .

Ω = all Boolean functions

$$\Omega_{a*} = \{ f \mid f(0, \dots, 0) = a \} \quad \Omega_{a*} \cup C$$

$$\Omega_{*b} = \{ f \mid f(1, \dots, 1) = b \} \quad \Omega_{*b} \cup C$$

$$\Omega_{\leq} = \{ f \mid f(0, \dots, 0) \leq f(1, \dots, 1) \}$$

$$\Omega_{\geq} = \{ f \mid f(0, \dots, 0) \geq f(1, \dots, 1) \}$$

M = monotone functions

\bar{M} = anti-monotone functions

$$R = \{ f \mid \forall \mathbf{a} \in \text{dom}(f) f(\mathbf{a}) = f(\bar{\mathbf{a}}) \} \quad (\text{reflexive functions})$$

Moreover, intersections of (J, MU_{01}^k) -clonoids are (J, MU_{01}^k) -clonoids.

Definition

Let $C \in \{\Omega_{*1}, \Omega_{1*}, M, \overline{M}, R\}$. The C -closure of f , denoted by f^C , is the least majorant g of f such that $g \in C$.

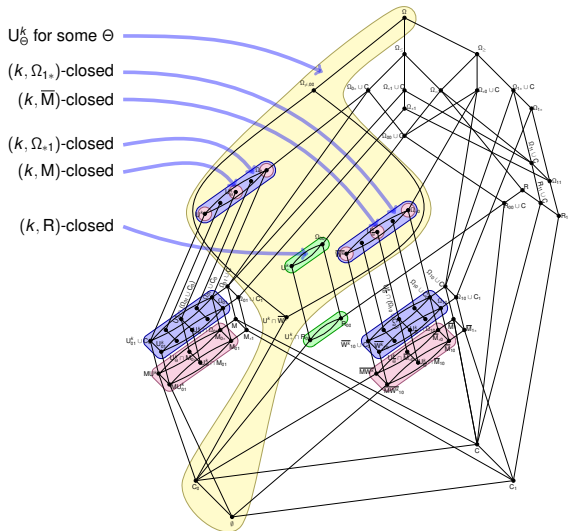
Definition

Let $\Phi \subseteq \Omega$, and let $C \in \{\Omega_{*1}, \Omega_{1*}, M, \overline{M}, R\}$. The set Φ is (k, C) -closed if for every $\varphi \in \Phi$ and for every $T \subseteq (\varphi^C)^{-1}(1)$ with $|T| \leq k$, we have $\chi_T \in \Phi$.

Proposition

Let $k \in \mathbb{N} \cup \{\infty\}$, $C \in \{\Omega_{*1}, \Omega_{1*}, M, \overline{M}, R\}$, and $\Theta \in \text{Id}(\Omega^{[\leq k]})$. Then $U_{\Theta}^k \cap C = (U_{\Psi}^k)^C = U_{\Psi}^k \cap C$, where Ψ is the largest (k, C) -closed subset of Θ .

(J, MU_{01}^k) -clonoids



A schematic Hasse diagram of the poset of (J, MU_{01}^k) -clonoids.

Theorem

For $k \in \mathbb{N}$ with $k \geq 2$, the (J, MU_{01}^k) -clonoids are the following:

- 1 U_{Θ}^k for each $\Theta \in \text{Id}(\Omega^{[\leq k]})$,
- 2 $U_{\Theta}^k \cap (\Omega_{01} \cup C_0)$ and $U_{\Theta}^k \cap \Omega_{01}$ for each nonempty $\Theta \in \text{Id}(\Omega_{\Omega_{*1}}^{[\leq k]})$,
- 3 $U_{\Theta}^k \cap (\Omega_{10} \cup C_0)$ and $U_{\Theta}^k \cap \Omega_{10}$ for each nonempty $\Theta \in \text{Id}(\Omega_{\Omega_{1*}}^{[\leq k]})$,
- 4 $U_{\Theta}^k \cap M_{0*}$ and $U_{\Theta}^k \cap M_{01}$ for each nonempty $\Theta \in \text{Id}(\Omega_M^{[\leq k]})$,
- 5 $U_{\Theta}^k \cap \bar{M}_{*0}$ and $U_{\Theta}^k \cap \bar{M}_{10}$ for each nonempty $\Theta \in \text{Id}(\Omega_{\bar{M}}^{[\leq k]})$,
- 6 $U_{\Theta}^k \cap R_{00}$ for each nonempty $\Theta \in \text{Id}(\Omega_R^{[\leq k]})$,
- 7 $\Omega_{\leq}, \Omega_{\geq}, \Omega_{=}, \Omega_{0*} \cup C, \Omega_{*0} \cup C, \Omega_{1*} \cup C, \Omega_{*1} \cup C, \Omega_{00} \cup C, \Omega_{01} \cup C, \Omega_{10} \cup C, \Omega_{11} \cup C, \Omega_{01} \cup C_1, \Omega_{10} \cup C_1, \Omega_{1*}, \Omega_{*1}, \Omega_{11}, M, M_{*1}, \bar{M}, \bar{M}_{1*}, R, R_{00} \cup C, R_{11} \cup C, R_{11}, C, C_1$.

Sparks's theorem deals with (J, C) -clonoids.

How about (C, J) -clonoids?

Let $f: A^n \rightarrow B$, $g: A^m \rightarrow B$. Let C be a clone on A . We say that f is a **C-minor** of g , and we write $f \leq_C g$, if $f \in \{g\}C$, i.e., there exist $h_1, \dots, h_m \in C^{(n)}$ such that $f = g(h_1, \dots, h_m)$.

The C -minor relation \leq_C is a quasiorder on \mathcal{F}_{AB} .

Note that $\{g\}C = J_B(\{g\}C) = \langle g \rangle_{(C, J_B)}$

Therefore, $f \leq_C g$ if and only if $f \in \langle g \rangle_{(C, J_B)}$.

Consequently, (C, J_B) -clonoids are precisely the downsets of the C -minor quasiorder \leq_C .

E. LEHTONEN, Descending chains and antichains of the unary, linear, and monotone subfunction relations, *Order* **23** (2006) 129–142.

E. LEHTONEN, Á. SZENDREI, Equivalence of operations with respect to discriminator clones, *Discrete Math.* **309** (2009) 673–685.

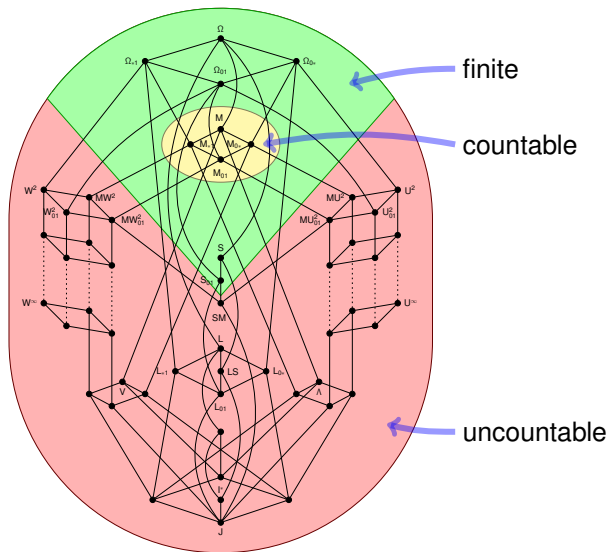
E. LEHTONEN, J. NEŠETŘIL, Minor of Boolean functions with respect to clique functions and hypergraph homomorphisms, *European J. Combin.* **31** (2010) 1981–1995.

Theorem

Let C be a clone on $\{0, 1\}$, and let J be the clone of projections on $\{0, 1\}$. Then the following statements hold.

- 1 $\mathcal{L}_{(C, J)}$ is finite if and only if C contains the discriminator function.*
- 2 $\mathcal{L}_{(C, J)}$ is countably infinite if and only if $\langle \wedge, \vee \rangle \subseteq C \subseteq \langle \wedge, \vee, 0, 1 \rangle$.*
- 3 $\mathcal{L}_{(C, J)}$ has the cardinality of the continuum otherwise.*

Cardinality of the lattice of (C, J) -clonoids



Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | ? | ? | ? | ? | ? | ? | C | F |
| I | ? | ? | ? | ? | ? | ? | C | F |
| I^* | ? | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | ? | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | ? | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | ? | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | ? | ? | ? | ? | ? | ? | F | F |
| V, Λ | ? | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | ? | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | ? | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | ? | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | ? | ? | ? | ? | ? | ? | F | F |
| L_{01} | ? | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | ? | ? | ? | ? | ? | ? | C | F |
| LS | ? | ? | ? | ? | ? | ? | C | F |
| L | ? | ? | ? | ? | ? | ? | C | F |
| SM | ? | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | ? | ? | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | ? | ? | ? | ? | ? | ? | F | F |

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | ? | ? | ? | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | ? | ? | ? | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | ? | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | F | ? | ? | ? | ? | ? | F | F |

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | ? | ? | ? | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | ? | ? | ? | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | ? | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Labeled posets

For $k \in \mathbb{N}$, a **k -poset** is a structure $\mathbf{P} = (P, \leq, c)$, where (P, \leq) is a poset and $c: P \rightarrow [0, k - 1]$ is a **labeling**.

Let $\mathbf{P} = (P, \leq, c)$ and $\mathbf{P}' = (P', \leq', c')$ be k -posets. A mapping $h: P \rightarrow P'$ is a **homomorphism** of \mathbf{P} to \mathbf{P}' if h preserves both the order and the labels, i.e., $h(x) \leq' h(y)$ whenever $x \leq y$, and $c = c' \circ h$.

The homomorphism order of k -posets ($k \geq 2$) has a very rich structure – it is universal for the class of countable posets.

This is the case for k -lattices for $k \geq 3$ as well. However, the homomorphism order of 2-lattices is rather easy to describe.

Clones of monotone functions

Let (A, \leq) be a poset.

Denote by \perp and \top the least and the greatest elements of (A, \leq) , provided that such elements exist.

The following sets are clones on A .

- M monotone functions with respect to \leq
- M_{\perp} monotone \perp -preserving functions
- M_{\top} monotone \top -preserving functions
- $M_{\perp\top}$ monotone \perp - and \top -preserving functions

Monotone minors

Let (A, \leq) be a poset. Assume $A = [0, k - 1]$.
We associate with each $f: A^n \rightarrow A$ the k -poset
 $P(f, \leq) := ((A, \leq)^n, f) = (A^n, \leq^n, f)$.

Proposition

Let (A, \leq) be a poset.

- 1 $f \leq_M g$ if and only if there exists a homomorphism of $P(f, \leq)$ to $P(g, \leq)$.
- 2 If (A, \leq) has a least element \perp , then $f \leq_{M_\perp} g$ if and only if there exists a \perp -preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.
- 3 If (A, \leq) has a greatest element \top , then $f \leq_{M_\top} g$ if and only if there exists a \top -preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.
- 4 If (A, \leq) has both a least element \perp and a greatest element \top , then $f \leq_{M_{\perp\top}} g$ if and only if there exists a \perp - and \top -preserving homomorphism of $P(f, \leq)$ to $P(g, \leq)$.

Proof.

Let C be one of $M, M_{\perp}, M_{\top}, M_{\perp\top}$.

Assume $f \leq_C g$. Then $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in C$. Clearly, $h = (h_1, \dots, h_m)$ is an order-preserving map from $(A, \leq)^n$ to $(A, \leq)^m$ and $f = g \circ h$, so h is a homomorphism of $P(f, \leq)$ to $P(g, \leq)$. Moreover, if $C \subseteq M_{\perp}$, then $h(\perp, \dots, \perp) = \perp$, so h is \perp -preserving. Similarly, if $C \subseteq M_{\top}$, then $h(\top, \dots, \top) = \top$, so h is \top -preserving.

Conversely, assume that there exists a homomorphism h of $P(f, \leq)$ to $P(g, \leq)$. Then $f = g \circ h$, and in $h = (h_1, \dots, h_m)$ each component function h_i is in M . Moreover, if h is \perp -preserving, then each h_i is in M_{\perp} ; similarly, if h is \top -preserving, then each h_i is in M_{\top} . Therefore, $f \leq_C g$. □

Alternating k -chains

A k -chain (A, \leq, c) with $A = \{a_0, a_1, \dots, a_d\}$, $a_0 < a_1 < \dots < a_d$ is **alternating** if $c(a_i) \neq c(a_{i+1})$ for all $i \in [0, d-1]$.

The number d is the **length** of this chain.

Note that an alternating 2-chain is uniquely determined, up to isomorphism, by its length and the label $c(a_0)$ of its least element; we have $c(a_i) = c(a_0)$ if and only if $i \equiv 0 \pmod{2}$.

For $d \in \mathbb{N}$ and $a \in \{0, 1\}$, denote by C_a^d the alternating 2-chain of length d with $c(a_0) = a$.

Proposition

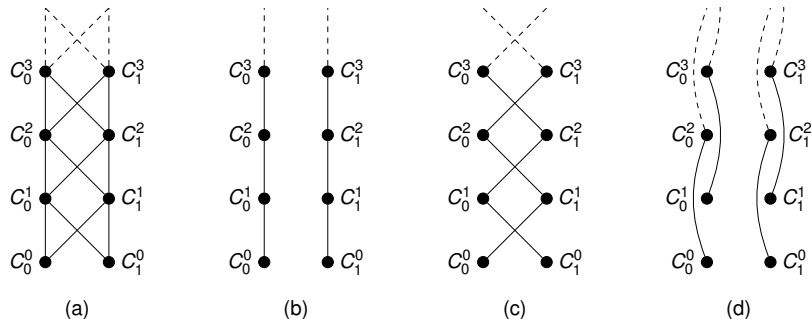
Let $d, e \in \mathbb{N}$ and $a, b \in \{0, 1\}$.

- 1 There exists a homomorphism of C_a^d to C_b^e if and only if $d < e$ or $(d, a) = (e, b)$.
- 2 There exists a \perp -preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$ and $a = b$.
- 3 There exists a \top -preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$ and $a + d \equiv b + e \pmod{2}$.
- 4 There exists a \perp - and \top -preserving homomorphism of C_a^d to C_b^e if and only if $d \leq e$, $a = b$, and $d \equiv e \pmod{2}$.

Proof.

Straightforward verification. □

Homomorphism orders of 2-chains



Homomorphism orders of 2-chains:

- (a) unrestricted,
- (b) \perp -preserving,
- (c) \top -preserving, and
- (d) \perp - and \top -preserving homomorphisms.

Proposition (Cf. Kosub, Wagner)

Every 2-poset with a least element (or with a greatest element) is homomorphically equivalent to its longest alternating 2-chain.

Proof.

Omitted from this presentation. □

S. KOSUB, K. W. WAGNER, The Boolean hierarchy of NP-partitions, in: H. Reichel, S. Tison (Eds.), STACS 2000, 17th Annual Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Comput. Sci., vol. 1770, Springer-Verlag, Berlin, 2000, pp. 157–168.

S. KOSUB, K. W. WAGNER, The Boolean hierarchy of NP-partitions, *Inform. and Comput.* **206** (2008) 538–568.

Alternation number of a Boolean function

Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$.

$P(k, \leq) := (\{0, 1\}^n, \leq^n, f)$

The **alternation number** of f , denoted by $\text{Alt}(f)$, is the length of the longest alternating 2-chain in $P(f, \leq)$, where \leq is the chain $0 < 1$.

Putting together the previous results, we obtain:

Proposition

Let $f, g \in \Omega$, and let $k := \text{Alt}(f)$, $\ell := \text{Alt}(g)$, $a := f(\mathbf{0})$, $b := g(\mathbf{0})$.

- 1 $f \leq_M g$ if and only if $k < \ell$ or $(k, a) = (\ell, b)$.
- 2 $f \leq_{M_{0*}} g$ if and only if $k \leq \ell$ and $a = b$.
- 3 $f \leq_{M_{*1}} g$ if and only if $k \leq \ell$ and $a + k \equiv b + \ell \pmod{2}$.
- 4 $f \leq_{M_{01}} g$ if and only if $k \leq \ell$, $a = b$, and $k \equiv \ell \pmod{2}$.

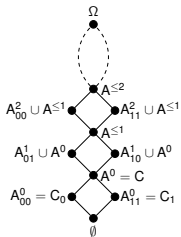
$$A^k := \{f \in \Omega \mid \text{Alt}(f) = k\} \quad A^{\leq k} := \{f \in \Omega \mid \text{Alt}(f) \leq k\}$$

(C, J) -clonoids for $C \in \{M_{01}, M_{0*}, M_{*1}, M\}$

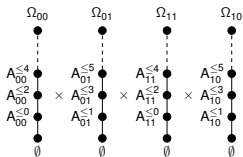
Theorem

- 1 The (M, J) -clonoids are the sets $\emptyset, \Omega, A_{00}^0 = C_0, A_{11}^0 = C_1, A^{\leq k}, A_{0*}^{k+1} \cup A^{\leq k}$, and $A_{1*}^{k+1} \cup A^{\leq k}$ for $k \in \mathbb{N}$.
- 2 The (M_{0*}, J) -clonoids are the sets of the form $A \cup B$, where
$$A \in \{\emptyset, \Omega_{0*}\} \cup \{A_{0*}^{\leq k} \mid k \in \mathbb{N}\}, \quad B \in \{\emptyset, \Omega_{1*}\} \cup \{A_{1*}^{\leq k} \mid k \in \mathbb{N}\}.$$
- 3 The (M_{*1}, J) -clonoids are the sets of the form $A \cup B$, where
$$A \in \{\emptyset, \Omega_{*0}\} \cup \{A_{*0}^{\leq k} \mid k \in \mathbb{N}\}, \quad B \in \{\emptyset, \Omega_{*1}\} \cup \{A_{*1}^{\leq k} \mid k \in \mathbb{N}\}.$$
- 4 The (M_{01}, J) -clonoids are the sets of the form $A \cup B \cup C \cup D$, where
$$A \in \{\emptyset, \Omega_{00}\} \cup \{A_{00}^{\leq k} \mid k \in \mathbb{N}\}, \quad B \in \{\emptyset, \Omega_{01}\} \cup \{A_{01}^{\leq k} \mid k \in \mathbb{N}\},$$
$$C \in \{\emptyset, \Omega_{11}\} \cup \{A_{11}^{\leq k} \mid k \in \mathbb{N}\}, \quad D \in \{\emptyset, \Omega_{10}\} \cup \{A_{10}^{\leq k} \mid k \in \mathbb{N}\}.$$

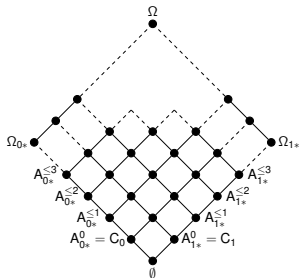
(C, J) -clonoids for $C \in \{M_{01}, M_{0*}, M_{*1}, M\}$



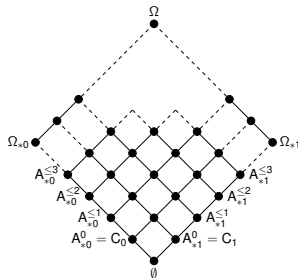
(M, J) -clonoids



(M_{01}, J) -clonoids



(M_{0*}, J) -clonoids



(M_{*1}, J) -clonoids

(C_1, C_2) -clonoids for $M_{01} \subseteq C_1 \subseteq M$, $C_2 \subseteq \Omega(1)$

Proposition

- 1 For clones $C_1 \in \{M, M_{0*}, M_{*1}, M_{01}\}$,
 - the (C_1, I_0) -clonoids are \emptyset and those (C_1, J) -clonoids K with $C_0 = A_{00}^0 \subseteq K$,
 - the (C_1, I_1) -clonoids are \emptyset and those (C_1, J) -clonoids K with $C_1 = A_{11}^0 \subseteq K$,
 - the (C_1, I) -clonoids are \emptyset and those (C_1, J) -clonoids K with $C = A^0 \subseteq K$.
- 2 For clones $C_1 \in \{M, M_{0*}, M_{*1}\}$ and $C_2 \in \{I^*, \Omega(1)\}$, the (C_1, C_2) -clonoids are the sets $\emptyset, \Omega, A^{\leq k}$, for $k \in \mathbb{N}$.
- 3 The (M_{01}, I^*) -clonoids are the sets of the form $A \cup B$, where $A \in \{\emptyset, \Omega_{=}\} \cup \{A_{00}^{\leq k} \cup A_{11}^{\leq k} \mid k \in \mathbb{N}\}$, $B \in \{\emptyset, \Omega_{\neq}\} \cup \{A_{01}^{\leq k} \cup A_{10}^{\leq k} \mid k \in \mathbb{N}\}$.
- 4 The $(M_{01}, \Omega(1))$ -clonoids are \emptyset and the sets of the form $A \cup B$, where $A \in \{\Omega_{=}\} \cup \{A_{00}^{\leq k} \cup A_{11}^{\leq k} \mid k \in \mathbb{N}\}$, $B \in \{\emptyset, \Omega_{\neq}\} \cup \{A_{01}^{\leq k} \cup A_{10}^{\leq k} \mid k \in \mathbb{N}\}$.

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | ? | ? | ? | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | ? | ? | ? | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | ? | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

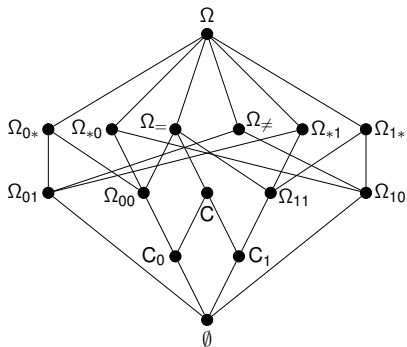
Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | ? | ? | ? | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | ? | ? | ? | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | C | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Example: (M_{01}, L_{01}) -clonoids

Proposition

*There are precisely 15 (M_{01}, L_{01}) -clonoids, and they are the following:
 $\Omega, \Omega_{0*}, \Omega_{1*}, \Omega_{*0}, \Omega_{*1}, \Omega_{=}, \Omega_{\neq}, \Omega_{00}, \Omega_{01}, \Omega_{10}, \Omega_{11}, C, C_0, C_1, \emptyset$.*

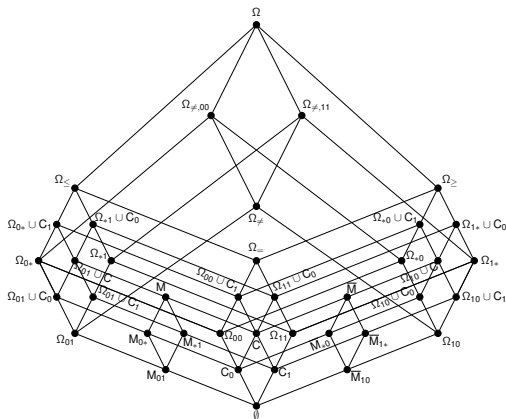


Example: (M_{01}, SM) -clonoids

Proposition

There are precisely 39 (M_{01}, SM) -clonoids, and they are the following:

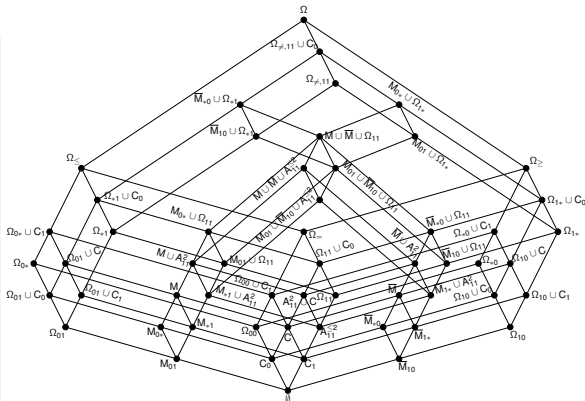
Ω , Ω_{\leq} , Ω_{\geq} , $\Omega_{\neq,00}$, $\Omega_{\neq,11}$,
 $\Omega_{0*} \cup C_1$, $\Omega_{*0} \cup C_1$,
 $\Omega_{1*} \cup C_0$, $\Omega_{*1} \cup C_0$, Ω_{0*} ,
 Ω_{*0} , Ω_{1*} , Ω_{*1} , $\Omega_{=}$, Ω_{\neq} ,
 $\Omega_{01} \cup C$, $\Omega_{10} \cup C$,
 $\Omega_{01} \cup C_0$, $\Omega_{10} \cup C_0$,
 $\Omega_{01} \cup C_1$, $\Omega_{10} \cup C_1$,
 $\Omega_{00} \cup C_1$, $\Omega_{11} \cup C_0$, Ω_{00} ,
 Ω_{11} , Ω_{01} , Ω_{10} , M , M_{0*} ,
 M_{*1} , M_{01} , \bar{M} , \bar{M}_{*0} , \bar{M}_{1*} ,
 \bar{M}_{10} , C , C_0 , C_1 , \emptyset .



Example: (M_{01}, V_{01}) -clonoids

Proposition

There are precisely 56 (M_{01}, V_{01}) -clonoids, and they are the following: Ω , $\Omega_{\neq,11} \cup C_0$, $\Omega_{\neq,11}$, Ω_{\geq} , Ω_{\leq} , $\Omega_{=}$, $\Omega_{0*} \cup C_1$, $\Omega_{*0} \cup C_1$, $\Omega_{1*} \cup C_0$, $\Omega_{*1} \cup C_0$, $\Omega_{01} \cup C$, $\Omega_{01} \cup C_0$, $\Omega_{01} \cup C_1$, $\Omega_{10} \cup C$, $\Omega_{10} \cup C_0$, $\Omega_{10} \cup C_1$, $\Omega_{00} \cup C_1$, $\Omega_{11} \cup C_0$, Ω_{0*} , Ω_{*0} , Ω_{1*} , Ω_{*1} , Ω_{01} , Ω_{10} , Ω_{00} , Ω_{11} , $M \cup \bar{M} \cup \Omega_{11}$, $M \cup \bar{M} \cup A_{11}^2$, $M_{01} \cup \bar{M}_{10} \cup \Omega_{11}$, $M_{01} \cup \bar{M}_{10} \cup A_{11}^{\leq 2}$, $M \cup A_{11}^2$, $M_{0*} \cup \Omega_{1*}$, $M_{0*} \cup \Omega_{11}$, $M_{*1} \cup A_{11}^2$, $\bar{M} \cup A_{11}^2$, $\bar{M}_{*0} \cup \Omega_{*1}$, $\bar{M}_{*0} \cup \Omega_{11}$, $\bar{M}_{1*} \cup A_{11}^2$, $M_{01} \cup \Omega_{1*}$, $M_{01} \cup \Omega_{11}$, $\bar{M}_{10} \cup \Omega_{*1}$, $\bar{M}_{10} \cup \Omega_{11}$, M , M_{0*} , M_{*1} , M_{01} , \bar{M} , \bar{M}_{*0} , \bar{M}_{1*} , \bar{M}_{10} , $A_{11}^2 \cup C$, $A_{11}^{\leq 2}$, C , C_0 , C_1 , \emptyset .



Proposition

There are precisely 56 (M_{01}, Λ_{01}) -clonoids, and they are the duals of the (M_{01}, V_{01}) -clonoids.

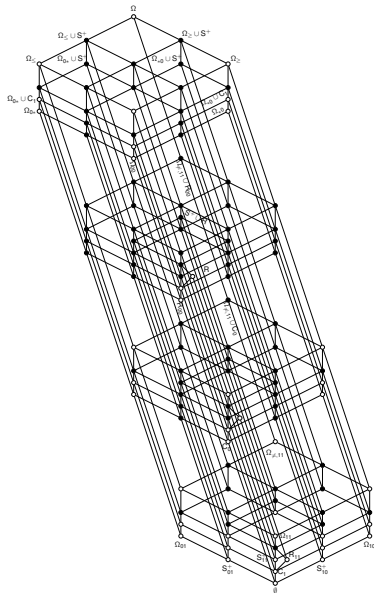
Example: (S_{01}, V_{01}) -clonoids

Theorem

There are precisely 123 (S_{01}, V_{01}) -clonoids. They are precisely the classes

$$\begin{aligned} &\Omega, \quad \Omega_{\leq} \cup S^+, \quad \Omega_{\geq} \cup S^+, \\ &\Omega_{\leq}, \quad \Omega_{\geq}, \quad \Omega_{0*} \cup S^+, \quad \Omega_{*0} \cup S^+, \\ &\Omega_{0*} \cup C_1, \quad \Omega_{*0} \cup C_1, \quad \Omega_{0*}, \quad \Omega_{*0}, \\ &\Omega_{\neq,11} \cup R_{00}, \quad \Omega_{\neq,11} \cup C_0, \quad \Omega_{\neq,11}, \\ &S^+ \cup R, \quad R, \end{aligned}$$

and their intersections.



Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | ? | ? | ? | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | ? | ? | ? | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | ? | ? | ? | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | ? | ? | ? | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | ? | ? | ? | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | C | ? | ? | ? | ? | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | F | F | F | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | F | F | F | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | F | F | F | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | F | F | F | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | F | F | F | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | C | F | F | F | F | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Theorem

For clones C_1 and C_2 such that $C_1 \subseteq K_1$ and $C_2 \subseteq K_2$ for some

$$(K_1, K_2) \in \{(\Omega(1), \Omega(1)), (\Omega(1), \wedge), (\Omega(1), \vee), (I^*, U^\infty), (I^*, W^\infty), (I_0, U^\infty), (I_1, U^\infty), (I_0, W^\infty), (I_1, W^\infty), (L, \Omega(1)), (\wedge, \wedge), (\wedge, \vee), (\vee, \wedge), (\vee, \vee), (\wedge, \Omega(1)), (\vee, \Omega(1)), (L, \wedge), (L_{0*}, U^\infty), (L_{*1}, U^\infty), (LS, U^\infty), (L, \vee), (L_{0*}, W^\infty), (L_{*1}, W^\infty), (LS, W^\infty)\}$$

there are an uncountable infinitude of (C_1, C_2) -clonoids.

E. LEHTONEN, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones. arXiv:2412.01107

E. LEHTONEN, Clonoids of Boolean functions with a linear source clone and a semilattice or 0- or 1-separating target clone. arXiv:2504.04481

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1, C_2)} = \langle T \rangle_{(C_1, C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2) -clonoids.

One of the following families of functions does the job in each case.

- $f_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (L, \wedge)$.

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1, C_2)} = \langle T \rangle_{(C_1, C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2) -clonoids.

One of the following families of functions does the job in each case.

- $f_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (L, \wedge)$.

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1, C_2)} = \langle T \rangle_{(C_1, C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2) -clonoids.

One of the following families of functions does the job in each case.

- $f_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n: \{0, 1\}^n \rightarrow \{0, 1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (L, \wedge)$.

Definition

For $n \in \mathbb{N}^+$, define $\beta_n: \{0, 1\}^n \rightarrow \{0, 1\}$ by the rule $\beta_n(\mathbf{a}) = 1$ if and only if $w(\mathbf{a}) \in \{1, 2, n\}$.

For $N \subseteq [n]$, let $B_N := \{\beta_n \mid n \in N\}$.

Lemma

Let N be the set of odd integers greater than 6. Let $S \subseteq N$ and $m \in N$. We have $\beta_m \in \langle B_S \rangle_{(L, \wedge)}$ if and only if $m \in S$.

Proof

Sufficiency.

If $m \in S$, then clearly $\beta_m \in B_S \subseteq \langle B_S \rangle_{(C_1, C_2)}$ for all clones C_1 and C_2 .

Claim

Let m and n be odd integers greater than 6, and assume that $\beta_m \leq \varphi = \beta_n \circ \mathbf{g}$, where $\mathbf{g} = (g_1, \dots, g_n): \{0, 1\}^m \rightarrow \{0, 1\}^n$ with $g_1, \dots, g_n \in \mathbf{L}$ such that $\mathbf{g}(\mathbf{0}) =: \mathbf{c} \in \beta_m^{-1}(0)$. Then $m = n$.

Proof of the Claim. A combinatorial argument with a sprinkle of algebra, too lengthy to present here.

Proof of the Lemma (continued)

Necessity.

Assume $\beta_m \in \langle B_S \rangle_{(L, \wedge)} = \wedge(B_S L)$.

Then $\beta_m = \bigwedge_{i=1}^k \gamma_i$ for some $k \in \mathbb{N}^+$, where, for $i \in [k]$, $\gamma_i = \beta_{n_i} \circ \mathbf{g}^i$, where $n_i \in S$ and $\mathbf{g}^i = (g_1^i, \dots, g_{n_i}^i)$ for some $g_1^i, \dots, g_{n_i}^i \in L$.

We have $\beta_m \leq \gamma_i$ for all $i \in [k]$, and for every $\mathbf{a} \in \beta_m^{-1}(0)$, there exists an $i \in [k]$ such that $\gamma_i(\mathbf{a}) = 0$.

In particular, there is a $j \in [k]$ such that $\gamma_j(\mathbf{0}) = 0$, that is, $\mathbf{g}^j(\mathbf{0}) \in \beta_{n_j}^{-1}(0)$.

By our Claim, $m = n_j$, so $m \in S$. □

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | ? | ? | ? | ? | ? | C | F |
| I | U | ? | ? | F | F | F | C | F |
| I^* | U | ? | ? | ? | ? | ? | C | F |
| $\Omega(1)$ | U | ? | ? | F | F | F | C | F |
| V_{01}, Λ_{01} | U | ? | ? | ? | ? | ? | F | F |
| V_{0*}, Λ_{*1} | U | ? | ? | F | F | F | F | F |
| V_{*1}, Λ_{0*} | U | ? | ? | ? | ? | ? | F | F |
| V, Λ | U | ? | ? | F | F | F | F | F |
| MU_{01}^k, MW_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| MU^k, MW^k | U | ? | ? | ? | ? | ? | F | F |
| U_{01}^k, W_{01}^k | U | ? | ? | ? | ? | ? | F | F |
| U^k, W^k | U | ? | ? | ? | ? | ? | F | F |
| L_{01} | U | ? | ? | ? | ? | ? | C | F |
| L_{0*}, L_{*1} | U | ? | ? | ? | ? | ? | C | F |
| LS | U | ? | ? | ? | ? | ? | C | F |
| L | U | ? | ? | F | F | F | C | F |
| SM | U | ? | ? | ? | ? | ? | F | F |
| $[M_{01}, M]$ | C | C | F | F | F | F | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, I]$ | $[I^*, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, MU^\infty]$ $[MW_{01}^\infty, MW^\infty]$ | U_{01}^∞ W_{01}^∞ | U^∞ W^∞ | $[L_{01}, L]$ | $\{SM, MU_{01}^k, MW_{01}^k\}, \Omega$ |
|------------------------|----------|--------------------|--|--|------------------------------------|--------------------------|---------------|--|
| J | U | U | U | U | U | U | C | F |
| I_0, I_1 | U | U | U | U | U | U | C | F |
| I | U | U | U | F | F | F | C | F |
| I^* | U | U | U | U | U | U | C | F |
| $\Omega(1)$ | U | U | U | F | F | F | C | F |
| V_{01}, Λ_{01} | U | U | U | U | U | U | F | F |
| V_{0*}, Λ_{*1} | U | U | U | F | F | F | F | F |
| V_{*1}, Λ_{0*} | U | U | U | U | U | U | F | F |
| V, Λ | U | U | U | F | F | F | F | F |
| MU_{01}^k, MW_{01}^k | U | U | U | U | U | U | F | F |
| MU^k, MW^k | U | U | U | U | U | U | F | F |
| U_{01}^k, W_{01}^k | U | U | U | U | U | U | F | F |
| U^k, W^k | U | U | U | U | U | U | F | F |
| L_{01} | U | U | U | U | U | U | C | F |
| L_{0*}, L_{*1} | U | U | U | U | U | U | C | F |
| LS | U | U | U | U | U | U | C | F |
| L | U | U | U | F | F | F | C | F |
| SM | U | U | U | U | U | U | F | F |
| $[M_{01}, M]$ | C | C | F | F | F | F | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F | F | F | F |

Theorem

Let C_1 and C_2 be clones on $\{0, 1\}$. Then the lattice $\mathcal{L}_{(C_1, C_2)}$ of (C_1, C_2) -clonoids is

① *finite if $C_1 \supseteq K_1$ and $C_2 \supseteq K_2$ for some*

$$(K_1, K_2) \in \{(J, MU_{01}^k) \mid 2 \leq k < \infty\} \cup \{(J, MW_{01}^k) \mid 2 \leq k < \infty\} \cup \\ \{(J, SM), (I, MU_{01}^\infty), (I, MW_{01}^\infty), (\Lambda_{01}, L_{01}), (V_{01}, L_{01}), \\ (V_{0*}, MU_{01}^\infty), (V_{0*}, MW_{01}^\infty), (\Lambda_{*1}, MU_{01}^\infty), (\Lambda_{*1}, MW_{01}^\infty), \\ (M_{01}, \Lambda_{01}), (M_{01}, V_{01}), (S_{01}, J)\};$$

② *countably infinite if $C_1 \in [J, L]$ and $C_2 \in [L_{01}, L]$ or $C_1 \in [M_{01}, M]$ and $C_2 \in [J, \Omega(1)]$;*

③ *uncountable otherwise.*

Cardinalities of (C_1, C_2) -clonoid lattices

| | $[J, \Omega(1)]$ | $[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$ | $[MU_{01}^\infty, U^\infty]$ $[MW_{01}^\infty, W^\infty]$ | $[L_{01}, L]$ | $2 \leq \ell < \infty$ $[\{SM, MU_{01}^\ell, MW_{01}^\ell\}, \Omega]$ |
|--|------------------|--|--|---------------|--|
| $[J, \{L_{0*}, L_{*1}, LS\}]$ | U | U | U | C | F |
| $[I, L]$ | U | U | F | C | F |
| $[\{\Lambda_{01}, V_{01}, SM\}, \{U^2, W^2\}]$ | U | U | U | F | F |
| $[\Lambda_{*1}, \Lambda], [V_{0*}, V]$ | U | U | F | F | F |
| $[M_{01}, M]$ | C | F | F | F | F |
| $[S_{01}, \Omega]$ | F | F | F | F | F |

- Generalize these results for clonoids on arbitrary (finite) base sets. (See the work by S. Kreinecker, S. Fioravanti, P. Mayr, P. Wynne.)
- Find necessary and sufficient conditions for the finiteness of the lattice of (C_1, C_2) -clonoids. (The “uniform generation by minors” by S. Fioravanti, M. Kompatscher, B. Rossi gives a sufficient condition.)
- Find simple relational descriptions for clonoids.

M. COUCEIRO, E. LEHTONEN, Stability of Boolean function classes with respect to clones of linear functions, *Order* **41** (2024) 15–64.

E. LEHTONEN, Majority-closed minions of Boolean functions, *Algebra Universalis* **85** (2024) Art. 6.

E. LEHTONEN, Near-unanimity-closed minions of Boolean functions, *Algebra Universalis* **86** (2025) Art. 2.

E. LEHTONEN, Clonoids of Boolean functions with a monotone or discriminator source clone. *Algebra Universalis* **87** (2026) Art. 4.

E. LEHTONEN, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones. *Internat. J. Algebra Comput.* **36**(1) (2026) 17–50.

E. LEHTONEN, Clonoids of Boolean functions with a linear source clone and a semilattice or 0- or 1-separating target clone. *Algebra Universalis*, to appear.

Thank you for your attention!