جامعة خليفة
Khalifa University

The cardinalities of clonoid lattices of Boolean functions

Erkko Lehtonen

Department of Mathematics
Khalifa University of Science and Technology
Abu Dhabi, United Arab Emirates

19th International Conference of Young Algebraists in Thailand

Department of Mathematics, Faculty of Science

Mahasarakham University

25–27 March 2026

Context

Clone theory is a subfield of universal algebra that focuses on the study of **clones**, sets of operations on a fixed set that include all projections and are closed under composition.

The set of all term operations of an algebra is a clone. In fact, every clone arises in this way.

It is often seen in universal algebra that many properties of an algebra do not depend so much on its fundamental operations but on its clone of term operations.

Clonoids are a generalization of clones that allows functions from powers of one set (or algebra) into another.

Minions (or **minor-closed classes**) are sets of functions that are closed under formation of **minors**. This is a special case of clonoids.

Context

The terminology we use is quite modern.

- The term “clone” was first used in the 1965 monograph of P. M. Cohn, who attributed it to Philip Hall.
- The term “clonoid” first appeared in a 2016 paper of E. Aichinger and P. Mayr.
- The term “minion” was coined and popularized by J. Opršal around the year 2018.

These concepts have nevertheless appeared in the literature much earlier under different names.

P. M. COHN, *Universal Algebra,* Harper & Row, New York, NY, 1965.

E. AICHINGER, P. MAYR, Finitely generated equational classes, *J. Pure Appl. Algebra* **220** (2016) 2816–2827.

J. BULÍN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, In: *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing* (*STOC '19*), *June 23–26, 2019, Phoenix, AZ, USA,* ACM, New York, NY, USA, pp. 602–613.

Functions, operations, projections

A **function** (**of several arguments**) from A to B is a mapping $f\colon A^n \to B$ for some $n \in \mathbb{N}$, called the **arity** of f.

If $A = B$, then we speak of **operations** on A.

Notation: $\mathcal{F}_{AB}^{(n)} := B^{A^n}$, $\mathcal{F}_{AB} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_{AB}^{(n)}$
$\mathcal{O}_A^{(n)} := \mathcal{F}_{AA}^{(n)}$, $\mathcal{O}_A := \mathcal{F}_{AA}$

The i-th n-ary **projection** on A is the operation $\mathrm{pr}_i^{(n)}\colon A^n \to A$, $(a_1, \dots, a_n) \mapsto a_i$.

Denote by J_A the set of all projections on A.

Functions, operations, projections

A **function** (**of several arguments**) from A to B is a mapping $f\colon A^n \to B$ for some $n \in \mathbb{N}$, called the **arity** of f.

If $A = B$, then we speak of **operations** on A.

Notation: $\mathcal{F}_{AB}^{(n)} := B^{A^n}$, $\mathcal{F}_{AB} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_{AB}^{(n)}$
$\mathcal{O}_{A}^{(n)} := \mathcal{F}_{AA}^{(n)}$, $\mathcal{O}_{A} := \mathcal{F}_{AA}$

The i-th n-ary **projection** on A is the operation $\mathrm{pr}_i^{(n)}\colon A^n \to A$, $(a_1, \dots, a_n) \mapsto a_i$.

Denote by J_A the set of all projections on A.

Composition of functions

Let $f \in \mathcal{F}_{BC}^{(n)}$, $g_1, \dots, g_n \in \mathcal{F}_{AB}^{(m)}$.
The **composition** of f with $g_1, \dots, g_n$ is the function $f(g_1, \dots, g_n) \in \mathcal{F}_{AC}^{(m)}$ defined by

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})).$$

Example

$$\begin{aligned}
f\colon \mathbb{R}^3 \to \mathbb{R}, &\quad f(x_1, x_2, x_3) = x_1 x_2^2 + 4x_3 \\
g_1\colon \mathbb{R}^2 \to \mathbb{R}, &\quad g_1(x_1, x_2) = x_1 + 2x_2 \\
g_2\colon \mathbb{R}^2 \to \mathbb{R}, &\quad g_2(x_1, x_2) = 3x_1 x_2 \\
g_3\colon \mathbb{R}^2 \to \mathbb{R}, &\quad g_3(x_1, x_2) = 5
\end{aligned}$$

$$\begin{aligned}
f(g_1, g_2, g_3)(x_1, x_2) &= (x_1 + 2x_2)(3x_1 x_2)^2 + 4 \cdot 5 \\
&= 9x_1^3 x_2^2 + 18x_1^2 x_2^3 + 20
\end{aligned}$$

Function class composition

Let $F \subseteq \mathcal{F}_{BC}$, $G \subseteq \mathcal{F}_{AB}$.

The **composition** of F with G, denoted by FG, is defined as

$$FG := \{\, f(g_1, \dots, g_n) \mid n, m \in \mathbb{N}_+,\ f \in F^{(n)},\ g_1, \dots, g_n \in G^{(m)} \,\}.$$

Although the composition of functions is associative, function class composition is not. We only have

$$(FG)H \subseteq F(GH).$$

Clones

A **clone** is a class $C \subseteq \mathcal{O}_A$ such that $\mathrm{J}_A \subseteq C$ and $CC \subseteq C$.

J_A denotes the class of all projections on A.

The set of all clones on A constitutes a closure system, i.e.,

- $\mathcal{O}_A$ is a clone,
- arbitrary intersections of clones are clones.

For $F \subseteq \mathcal{O}_A$, denote by $\langle F \rangle$ the clone generated by F.

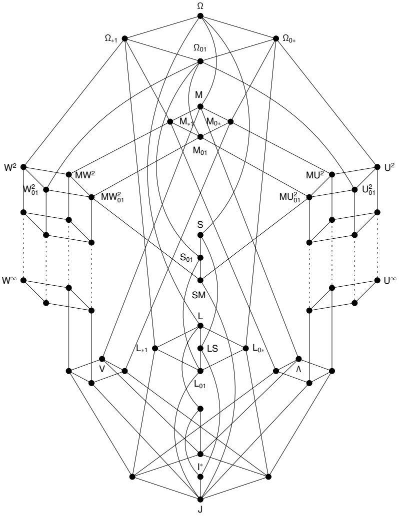
Clones

Examples of clones:

- $\mathcal{O}_A$
- J_A
- polynomial operations on $\mathbb{R}$
- for a partially ordered set $(A, \leq)$, monotone (order-preserving) operations
- essentially at most unary operations on A
- idempotent operations on A, i.e., operations f satisfying $f(a, \dots, a) = a$ for all $a \in A$
- for an algebra $\mathbf{A} = (A, (f_i)_{i \in I})$, the term operations of $\mathbf{A}$

Clones on $\{0, 1\}$

Post's lattice



E. POST, *The Two-Valued Iterative Systems of Mathematical Logic,* Annals of Mathematical Studies, no. 5, Princeton University Press, Princeton, 1941.

(C_1, C_2)-clonoids

Let $K \subseteq \mathcal{F}_{AB}$.

Let C_1 be a clone on A (the **source clone**), and
let C_2 be a clone on B (the **target clone**).

K is a (C_1, C_2)**-clonoid** (or a (C_1, C_2)**-stable** class)
if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Special case: $(\mathrm{J}_A, \mathrm{J}_B)$-clonoids are called **minions** or **minor-closed classes**.

(C_1, C_2)-clonoids

Let $K \subseteq \mathcal{F}_{AB}$.

Let C_1 be a clone on A (the **source clone**), and
let C_2 be a clone on B (the **target clone**).

K is a (C_1, C_2)**-clonoid** (or a (C_1, C_2)**-stable** class)
if $KC_1 \subseteq K$ and $C_2K \subseteq K$.

Special case: $(\mathrm{J}_A, \mathrm{J}_B)$-clonoids are called **minions** or **minor-closed classes**.

Examples of minions

Examples of minions:

- every clone
- polynomial operations on $\mathbb{R}$ of degree at most d
- for a partially ordered set $(A, \leq)$, order-reversing operations
- threshold functions
- for $a \in A$, $b \in B$, functions f satisfying $f(a, \dots, a) = b$
- constant operations
- $\emptyset$

Closure system of (C_1, C_2)-clonoids

For fixed clones C_1 and C_2, the set of all (C_1, C_2)-clonoids constitutes a closure system, i.e.,
– $\mathcal{F}_{AB}$ is a (C_1, C_2)-clonoid,
– arbitrary intersections of (C_1, C_2)-clonoids are (C_1, C_2)-clonoids.

The closure of a set $F \subseteq \mathcal{F}_{AB}$:
$\langle F \rangle_{(C_1,C_2)} = C_2(FC_1)$

Note that $(C_2F)C_1 \subseteq C_2(FC_1)$.
This holds as an equality if F is a minion.
(Couceiro and Foldes's Associativity Lemma)

M. COUCEIRO, S. FOLDES, Functional equations, constraints, definability of function classes, and functions of Boolean variables, *Acta Cybernet.* **18** (2007) 61–75.

Polymorphisms and invariant relations

Let $f \in \mathcal{O}_A^{(n)}$ and let $R \subseteq A^m$.

f **preserves** R (or f is a **polymorphism** of R, or R is an **invariant** of f), in symbols, $f \rhd R$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ ($i \in \{1, \dots, n\}$),

$$f \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{pmatrix} \in R.$$

Galois connection

$$\operatorname{Pol}(\mathcal{R}) := \{\, f \in \mathcal{O}_A \mid \forall R \in \mathcal{R} \colon f \rhd R \,\}$$
$$\operatorname{Inv}(\mathcal{F}) := \{\, R \in \mathcal{R}_A \mid \forall f \in \mathcal{F} \colon f \rhd R \,\}$$

The Galois closed sets of operations are the locally closed clones.

V. G. BODNARČUK, L. A. KALUŽNIN, V. N. KOTOV, B. A. ROMOV, Galois theory for Post algebras, I, II, *Kibernetika* **3** (1969) 1–10, **5** (1969) 1–9 (in Russian); English translation: *Cybernetics* **5** (1969) 243–252, 531–539.

D. GEIGER, Closed systems of functions and predicates, *Pacific J. Math.* **27** (1968) 95–100.

Polymorphisms and invariant relation pairs

Let $f \in \mathcal{F}_{AB}^{(n)}$ and let $R \subseteq A^m$, $S \subseteq B^m$.

f **preserves** (R, S), in symbols, $f \triangleright (R, S)$, if for all $(a_{1i}, \dots, a_{mi}) \in R$ $(i \in \{1, \dots, n\})$,

$$f \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \vdots \\ f(a_{m1}, \dots, a_{mn}) \end{pmatrix} \in S.$$

Galois connection

$$\mathrm{pPol}(\mathcal{R}) := \{\, f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{R} \colon f \triangleright (R, S) \,\}$$
$$\mathrm{pInv}(\mathcal{F}) := \{\, (R, S) \in \mathcal{R}_{AB} \mid \forall f \in \mathcal{F} \colon f \triangleright (R, S) \,\}$$

The Galois closed sets of functions are the locally closed minions.

N. Pippenger, Galois theory for minors of finite functions, *Discrete Math.* **254** (2002) 405–419.

M. Couceiro, S. Foldes, On closed sets of relational constraints and classes of functions closed under variable substitutions, *Algebra Universalis* **54** (2005) 149–165.

(C_1, C_2)-clonoids and invariant relation pairs

Theorem (Couceiro, Foldes (2009))

Let A and B be arbitrary nonempty sets, and let C_1 and C_2 be clones on A and B, respectively. Let $F \subseteq \mathcal{F}_{AB}$.
The following are equivalent.

1. *F is a locally closed (C_1, C_2)-clonoid.*
2. *F is definable by some set of relation pairs (R, S), where $R \in \operatorname{Inv} C_1$ and $S \in \operatorname{Inv} C_2$.*

M. Couceiro, S. Foldes, Function classes and relational constraints stable under compositions with clones, *Discuss. Math. Gen. Algebra Appl.* **29** (2009) 109–121.

Constraint satisfaction problems

Let $\mathbf{A} = (A, (R_i)_{i \in I})$ be a relational structure.

The **constraint satisfaction problem** (**CSP**) with template **A**, $\mathrm{CSP}(\mathbf{A})$, is the following decision problem:

Given a relational structure **C** of the same type as **A**, decide whether there exists a homomorphism $\mathbf{C} \to \mathbf{A}$.

The following are natural questions:

- Given a relational structure **A**, what is the computational complexity of $\mathrm{CSP}(\mathbf{A})$?
- For which CSP templates **A** is $\mathrm{CSP}(\mathbf{A})$ solvable in polynomial time (in P) / nondeterministically in polynomial time (in NP)?
- Given relational structures **A** and **B**, is $\mathrm{CSP}(\mathbf{A})$ polynomial-time reducible to $\mathrm{CSP}(\mathbf{B})$?

Constraint satisfaction problems

Let $\mathbf{A} = (A, (R_i)_{i \in I})$ and $\mathbf{B} = (B, (S_i)_{i \in I})$ be relational structures of the same type such that there exists a homomorphism $\mathbf{A} \to \mathbf{B}$.

The **promise constraint satisfaction problem** (**PCSP**) with template $(\mathbf{A}, \mathbf{B})$, $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$, is the following decision problem:

Given a relational structure **C** of the same type as **A** (and **B**), decide whether there exists a homomorphism $\mathbf{C} \to \mathbf{A}$ (answer "yes") or there does not exist a homomorphism $\mathbf{C} \to \mathbf{B}$ (answer "no").

(We don't care about the situation when there is no homomorphism $\mathbf{C} \to \mathbf{A}$ but there is a homomorphism $\mathbf{C} \to \mathbf{B}$. An algorithm solving $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is allowed to answer anything in this case.)

Constraint satisfaction problems

If $\mathrm{Pol}(\mathbf{A}_1) \subseteq \mathrm{Pol}(\mathbf{A}_2)$, then $\mathrm{CSP}(\mathbf{A}_2)$ is reducible to $\mathrm{CSP}(\mathbf{A}_1)$.

If $\mathrm{Pol}(\mathbf{A}_1)$ has a minion homomorphism to $\mathrm{Pol}(\mathbf{A}_2)$, then $\mathrm{CSP}(\mathbf{A}_2)$ is reducible to $\mathrm{CSP}(\mathbf{A}_1)$.

If $\mathrm{pPol}(\mathbf{A}_1, \mathbf{B}_1)$ has a minion homomorphism to $\mathrm{pPol}(\mathbf{A}_2, \mathbf{B}_2)$, then $\mathrm{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ is reducible to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$.

P. JEAVONS, On the algebraic structure of combinatorial problems, *Theor. Comput. Sci.* **200** (1998) 185–204.

A. BULATOV, P. JEAVONS, A. KROKHIN, Classifying the complexity of constraints using finite algebras, *SIAM J. Comput.* **34** (2005) 720–742.

L. BARTO, J. OPRŠAL, M. PINSKER, The wonderland of reflections, *Israel J. Math.* **223** (2018) 363–298.

J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing* (*STOC '19*), ACM, New York, 2019. pp. 602–613.

L. BARTO, J. BULIN, A. KROKHIN, J. OPRŠAL, Algebraic approach to promise constraint satisfaction, *J. ACM* **68** (2021) 1–66.

Question

Question

Given clones C_1 and C_2 on sets A and B, respectively, what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Question

Question

Given clones C_1 and C_2 on finite sets A and B, respectively, what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Question

Question

Given clones C_1 and C_2 on $\{0, 1\}$,
what are the (C_1, C_2)-clonoids?

What is the number of (C_1, C_2)-clonoids?

Cardinality of the lattice of clonoids

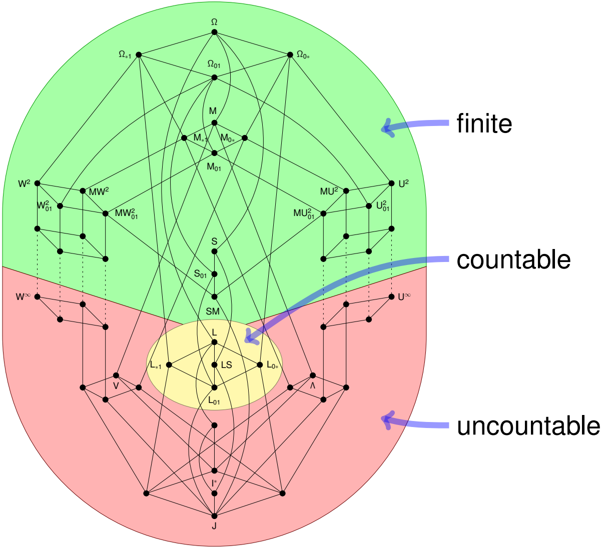
Theorem (Sparks 2019)

Let A be a finite set with $|A| > 1$, and let $B := \{0, 1\}$. Denote by J_A the clone of projections on A, and let C be a clone on B. Then the following statements hold.

1. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *is finite if and only if C contains a near-unanimity operation.*
2. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *is countably infinite if and only if C contains a Mal'cev operation but no majority operation.*
3. $\mathcal{L}_{(\mathrm{J}_A, C)}$ *has the cardinality of the continuum if and only if C contains neither a near-unanimity operation nor a Mal'cev operation.*

A. SPARKS, On the number of clonoids, *Algebra Universalis* **80**(4) (2019) Paper No. 53, 10 pp.

Cardinality of the lattice of (J_A, C)-clonoids



Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	?
I	?	?	?	?	?	?	?	?
I^*	?	?	?	?	?	?	?	?
$\Omega(1)$	?	?	?	?	?	?	?	?
V_{01}, Λ_{01}	?	?	?	?	?	?	?	?
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	?
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	?
V, Λ	?	?	?	?	?	?	?	?
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	?	?
MU^k, MW^k	?	?	?	?	?	?	?	?
U_{01}^k, W_{01}^k	?	?	?	?	?	?	?	?
U^k, W^k	?	?	?	?	?	?	?	?
L_{01}	?	?	?	?	?	?	?	?
L_{0*}, L_{*1}	?	?	?	?	?	?	?	?
LS	?	?	?	?	?	?	?	?
L	?	?	?	?	?	?	?	?
SM	?	?	?	?	?	?	?	?
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	?	?
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	?	?

Applying monotonicity of function class composition

Let C_1, C_1', C_2, C_2' be clones such that $C_1 \subseteq C_1'$ and $C_2 \subseteq C_2'$.

It follows from the monotonicity of function class composition that every (C_1', C_2')-clonoid is a (C_1, C_2)-clonoid.

(*Proof.* Assume K is a (C_1', C_2')-clonoid. Then $KC_1 \subseteq KC_1' \subseteq K$ and $C_2K \subseteq C_2'K \subseteq K$, so K is a (C_1, C_2)-clonoid. □)

Assume the (C_1, C_2)-clonoids are known.

In order to determine the (C_1', C_2')-clonoids, it suffices to identify them among the (C_1, C_2)-clonoids.

Applying monotonicity of function class composition

Let C_1, C'_1, C_2, C'_2 be clones such that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$.

It follows from the monotonicity of function class composition that every (C'_1, C'_2)-clonoid is a (C_1, C_2)-clonoid.

(*Proof.* Assume K is a (C'_1, C'_2)-clonoid. Then $KC_1 \subseteq KC'_1 \subseteq K$ and $C_2K \subseteq C'_2K \subseteq K$, so K is a (C_1, C_2)-clonoid. □)

Assume the (C_1, C_2)-clonoids are known.

In order to determine the (C'_1, C'_2)-clonoids, it suffices to identify them among the (C_1, C_2)-clonoids.

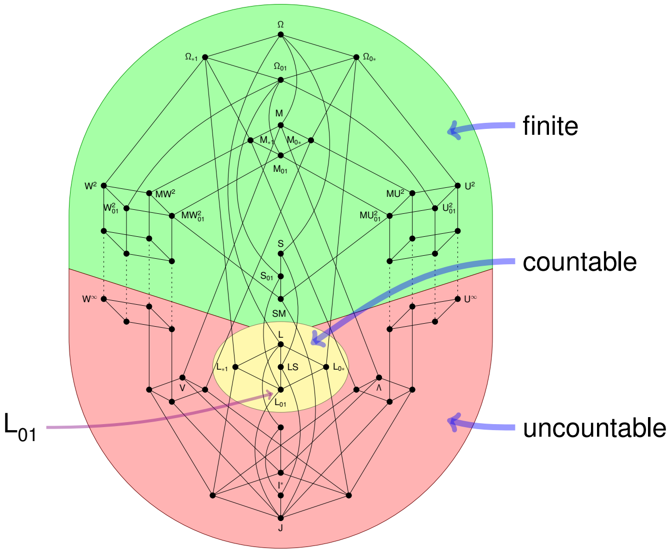
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	?
I	?	?	?	?	?	?	?	?
I^*	?	?	?	?	?	?	?	?
$\Omega(1)$	?	?	?	?	?	?	?	?
V_{01}, Λ_{01}	?	?	?	?	?	?	?	?
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	?
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	?
V, Λ	?	?	?	?	?	?	?	?
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	?	?
MU^k, MW^k	?	?	?	?	?	?	?	?
U_{01}^k, W_{01}^k	?	?	?	?	?	?	?	?
U^k, W^k	?	?	?	?	?	?	?	?
L_{01}	?	?	?	?	?	?	?	?
L_{0*}, L_{*1}	?	?	?	?	?	?	?	?
LS	?	?	?	?	?	?	?	?
L	?	?	?	?	?	?	?	?
SM	?	?	?	?	?	?	?	?
$[\mathrm{M}_{01}, \mathrm{M}]$	?	?	?	?	?	?	?	?
$[\mathrm{S}_{01}, \Omega]$	?	?	?	?	?	?	?	?

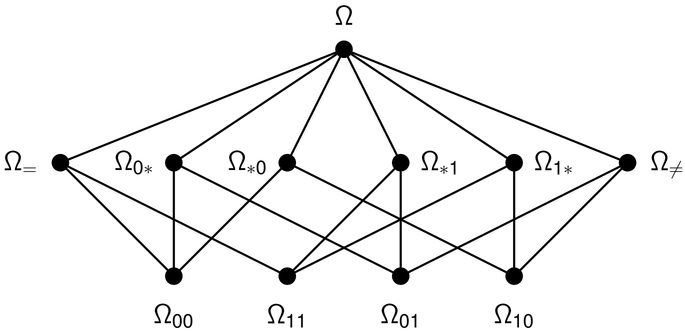
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	F
I	?	?	?	?	?	?	?	F
I^*	?	?	?	?	?	?	?	F
$\Omega(1)$	?	?	?	?	?	?	?	F
V_{01}, Λ_{01}	?	?	?	?	?	?	?	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	F
V, Λ	?	?	?	?	?	?	?	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	?	F
MU^k, MW^k	?	?	?	?	?	?	?	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	?	F
U^k, W^k	?	?	?	?	?	?	?	F
L_{01}	?	?	?	?	?	?	?	F
L_{0*}, L_{*1}	?	?	?	?	?	?	?	F
LS	?	?	?	?	?	?	?	F
L	?	?	?	?	?	?	?	F
SM	?	?	?	?	?	?	?	F
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	?	F
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	?	F

Cardinality of the lattice of (J_A, C)-clonoids



(J, L_{01})-clonoids



$$\begin{aligned}
\Omega &:= \text{all Boolean functions} \\
\Omega_{a*} &:= \{\, f \in \Omega \mid f(\mathbf{0}) = a \,\} \\
\Omega_{*b} &:= \{\, f \in \Omega \mid f(\mathbf{1}) = b \,\} \\
\Omega_{ab} &:= \Omega_{a*} \cap \Omega_{*b} \\
\Omega_{=} &:= \{\, f \in \Omega \mid f(\mathbf{0}) = f(\mathbf{1}) \,\} \\
\Omega_{\neq} &:= \{\, f \in \Omega \mid f(\mathbf{0}) \neq f(\mathbf{1}) \,\}
\end{aligned}$$

(J, L_{01})-clonoids

Zhegalkin polynomial of $f\colon \{0,1\}^n \to \{0,1\}$: $\displaystyle\sum_{S \in M_f} x_S, \quad x_S := \prod_{i \in S} x_i$

degree of f: $\deg(f) := \max_{S \in M_f} |S|$

characteristic rank of f: $\chi(f) := \deg(f + f^{\mathrm{n}}) + 1$

For $i, j \in \mathbb{N}$: $\mathrm{D}_i := \{\, f \in \Omega \mid \deg(f) \leq i \,\}$ $\mathrm{X}_j := \{\, f \in \Omega \mid \chi(f) \leq j \,\}$

Lemma (Selezneva, Bukhman (2016))

A Boolean function is reflexive if and only if $\chi(f) = 0$.

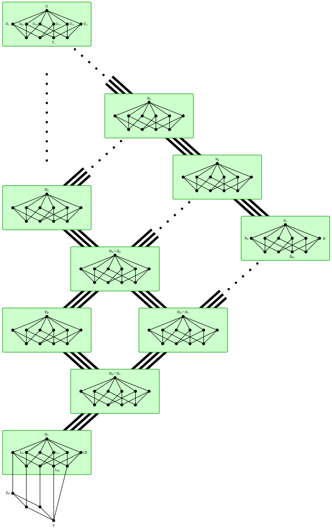
A Boolean function is self-dual if and only if $f + x_1$ is reflexive.

A Boolean function is self-dual if and only if f is odd and $\chi(f) = 1$.

S. N. SELEZNEVA, A. V. BUKHMAN, Polynomial-time algorithms for checking some properties of Boolean functions given by polynomials, *Theory Comput. Syst.* **58** (2016) 383–391.

(J, L_{01})-clonoids

$\aleph_0$ classes



M. Couceiro, E. Lehtonen, Stability of Boolean function classes with respect to clones of linear functions, *Order* **41** (2024) 15–64.

Proving that $\mathcal{L}_{(C_1,C_2)}$ is finite or countably infinite

A typical proof of a result describing $\mathcal{L}_{(C_1,C_2)}$ has three parts:

1. *Find the (C_1, C_2)-clonoids.*

   This requires some creativity and iterations when the 3rd step doesn't seem to work.

2. *Show that these classes are indeed (C_1, C_2)-clonoids.*

   This is straightforward verification. It suffices to do this for the meet-irreducible classes.

3. *Show that there are no further (C_1, C_2)-clonoids.*

   This is hard work. We show that for each class K is generated by any subset of K that is not contained in any lower cover of K (as suggested by the given list of classes).

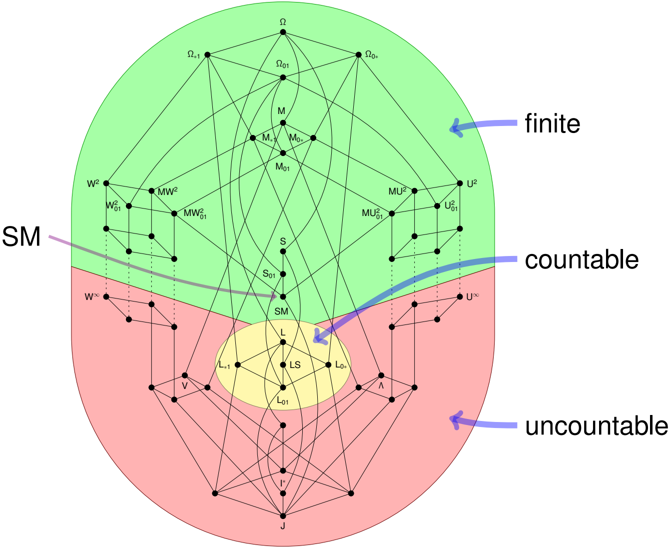
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	?	F
I	?	?	?	?	?	?	?	F
I^*	?	?	?	?	?	?	?	F
$\Omega(1)$	?	?	?	?	?	?	?	F
V_{01}, Λ_{01}	?	?	?	?	?	?	?	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	?	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	?	F
V, Λ	?	?	?	?	?	?	?	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	?	F
MU^k, MW^k	?	?	?	?	?	?	?	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	?	F
U^k, W^k	?	?	?	?	?	?	?	F
L_{01}	?	?	?	?	?	?	?	F
L_{0*}, L_{*1}	?	?	?	?	?	?	?	F
LS	?	?	?	?	?	?	?	F
L	?	?	?	?	?	?	?	F
SM	?	?	?	?	?	?	?	F
$[\mathrm{M}_{01}, \mathrm{M}]$	?	?	?	?	?	?	?	F
$[\mathrm{S}_{01}, \Omega]$	?	?	?	?	?	?	?	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^{\infty}, \mathsf{MU}^{\infty}]$ $[\mathsf{MW}_{01}^{\infty}, \mathsf{MW}^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	C	F
I	?	?	?	?	?	?	C	F
I^*	?	?	?	?	?	?	C	F
$\Omega(1)$	?	?	?	?	?	?	C	F
V_{01}, Λ_{01}	?	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	F	F
V, Λ	?	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	F	F
MU^k, MW^k	?	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	F	F
U^k, W^k	?	?	?	?	?	?	F	F
L_{01}	?	?	?	?	?	?	C	F
L_{0*}, L_{*1}	?	?	?	?	?	?	C	F
LS	?	?	?	?	?	?	C	F
L	?	?	?	?	?	?	C	F
SM	?	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	F	F

(J, SM)-clonoids



(J, SM)-clonoids

$$\Omega$$

$$\Omega_{\leq} := \{ f \in \Omega \mid f(\mathbf{0}) \leq f(\mathbf{1}) \}$$

$$\Omega_{\geq} := \{ f \in \Omega \mid f(\mathbf{0}) \geq f(\mathbf{1}) \}$$

$$\Omega_{\neq,00} := \Omega_{\neq} \cup \Omega_{00}$$

$$\Omega_{\neq,11} := \Omega_{\neq} \cup \Omega_{11}$$

$$\Omega_{0*} \cup \mathrm{C} \quad \Omega_{*1} \cup \mathrm{C} \quad \Omega_{*0} \cup \mathrm{C} \quad \Omega_{1*} \cup \mathrm{C}$$

$$\mathrm{S}^{-} := \{ f \in \Omega \mid \forall \mathbf{a} \colon f(\mathbf{a}) \wedge f(\overline{\mathbf{a}}) = 0 \}$$ (minorant of self-dual)

$$\mathrm{S}^{+} := \{ f \in \Omega \mid \forall \mathbf{a} \colon f(\mathbf{a}) \vee f(\overline{\mathbf{a}}) = 1 \}$$ (majorant of self-dual)

$$\mathrm{M} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \leq f(\mathbf{b}) \}$$ (monotone)

$$\overline{\mathrm{M}} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \geq f(\mathbf{b}) \}$$ (antitone)

$$\mathrm{U}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon f(\mathbf{a}) = f(\mathbf{b}) = 1 \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$$ (1-sep. of rank 2)

$$\mathrm{W}^2 := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon f(\mathbf{a}) = f(\mathbf{b}) = 0 \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$$ (0-sep. of rank 2)

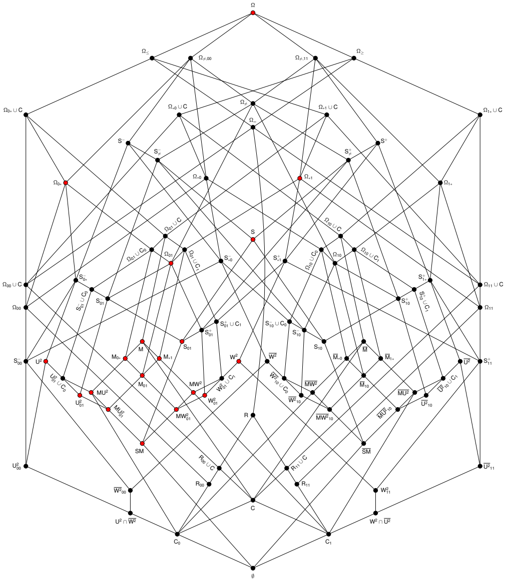
$$\overline{\mathrm{U}^2} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon f(\mathbf{a}) = f(\mathbf{b}) = 0 \Rightarrow \mathbf{a} \wedge \mathbf{b} \neq \mathbf{0} \}$$

$$\overline{\mathrm{W}^2} := \{ f \in \Omega \mid \forall \mathbf{a}, \mathbf{b} \colon f(\mathbf{a}) = f(\mathbf{b}) = 1 \Rightarrow \mathbf{a} \vee \mathbf{b} \neq \mathbf{1} \}$$

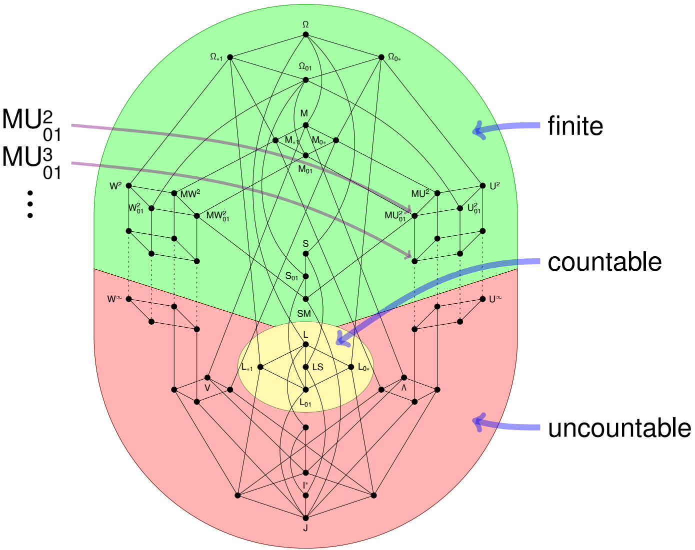
$$\mathrm{R} := \{ f \in \Omega \mid \forall \mathbf{a} \colon f(\mathbf{a}) = f(\overline{\mathbf{a}}) \}$$ (reflexive)

(J, SM)-clonoids

93 classes

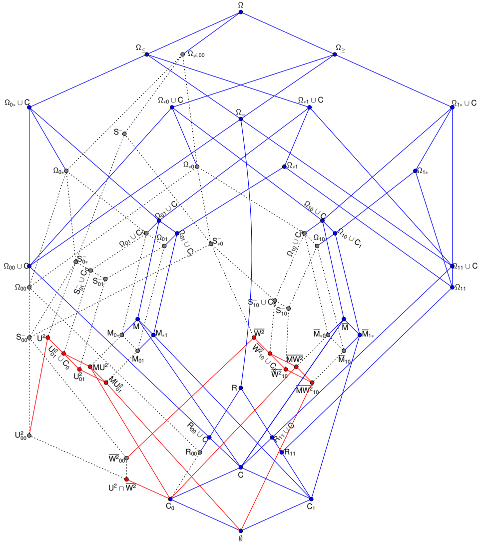


$(\mathrm{J}, \mathrm{MU}^k_{01})$-clonoids, $k \geq 2$

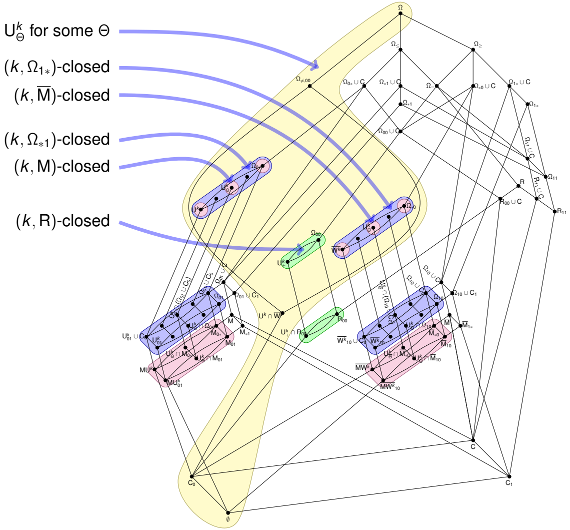


(J, MU^2_{01})-clonoids

63 classes



$(\mathrm{J}, \mathrm{MU}^k_{01})$-clonoids



A schematic Hasse diagram of the poset of $(\mathrm{J}, \mathrm{MU}^k_{01})$-clonoids.

$(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids

Theorem

For $k \in \mathbb{N}$ with $k \geq 2$, the $(\mathrm{J}, \mathrm{MU}_{01}^{k})$-clonoids are the following:

1. U_{Θ}^{k} *for each* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}^{[\leq k]})$,
2. $\mathrm{U}_{\Theta}^{k} \cap (\Omega_{01} \cup \mathrm{C}_0)$ *and* $\mathrm{U}_{\Theta}^{k} \cap \Omega_{01}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\Omega_{*1}}^{[\leq k]})$,
3. $\mathrm{U}_{\Theta}^{k} \cap (\Omega_{10} \cup \mathrm{C}_0)$ *and* $\mathrm{U}_{\Theta}^{k} \cap \Omega_{10}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\Omega_{1*}}^{[\leq k]})$,
4. $\mathrm{U}_{\Theta}^{k} \cap \mathrm{M}_{0*}$ *and* $\mathrm{U}_{\Theta}^{k} \cap \mathrm{M}_{01}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\mathrm{M}}^{[\leq k]})$,
5. $\mathrm{U}_{\Theta}^{k} \cap \overline{\mathrm{M}}_{*0}$ *and* $\mathrm{U}_{\Theta}^{k} \cap \overline{\mathrm{M}}_{10}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\overline{\mathrm{M}}}^{[\leq k]})$,
6. $\mathrm{U}_{\Theta}^{k} \cap \mathrm{R}_{00}$ *for each nonempty* $\Theta \in \mathrm{Id}(\boldsymbol{\Omega}_{\mathrm{R}}^{[\leq k]})$,
7. $\Omega_{\leq}$, $\Omega_{\geq}$, $\Omega_{=}$, $\Omega_{0*} \cup \mathrm{C}$, $\Omega_{*0} \cup \mathrm{C}$, $\Omega_{1*} \cup \mathrm{C}$, $\Omega_{*1} \cup \mathrm{C}$, $\Omega_{00} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{11} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}_1$, Ω_{1*}, Ω_{*1}, Ω_{11}, M, M_{*1}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{1*}$, R, $\mathrm{R}_{00} \cup \mathrm{C}$, $\mathrm{R}_{11} \cup \mathrm{C}$, R_{11}, C, C_1.

(C, J)-clonoids

Sparks's theorem deals with (J, C)-clonoids.

How about (C, J)-clonoids?

C-minors

Let $f\colon A^n \to B$, $g\colon A^m \to B$. Let C be a clone on A. We say that f is a **C-minor** of g, and we write $f \leq_C g$, if $f \in \{g\}C$, i.e., there exist $h_1, \dots, h_m \in C^{(n)}$ such that $f = g(h_1, \dots, h_m)$.

The C-minor relation $\leq_C$ is a quasiorder on $\mathcal{F}_{AB}$.

Note that $\{g\}C = \mathrm{J}_B(\{g\}C) = \langle g \rangle_{(C,\mathrm{J}_B)}$
Therefore, $f \leq_C g$ if and only if $f \in \langle g \rangle_{(C,\mathrm{J}_B)}$.

Consequently, (C, J_B)-clonoids are precisely the downsets of the C-minor quasiorder $\leq_C$.

E. LEHTONEN, Descending chains and antichains of the unary, linear, and monotone subfunction relations, *Order* **23** (2006) 129–142.

E. LEHTONEN, Á. SZENDREI, Equivalence of operations with respect to discriminator clones, *Discrete Math.* **309** (2009) 673–685.

E. LEHTONEN, J. NEŠETŘIL, Minor of Boolean functions with respect to clique functions and hypergraph homomorphisms, *European J. Combin.* **31** (2010) 1981–1995.

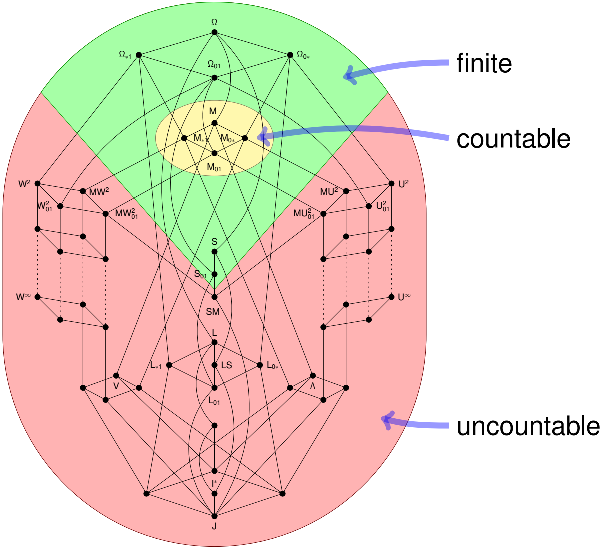
Cardinality of the lattice of (C, J)-clonoids

Theorem

Let C be a clone on $\{0, 1\}$, and let J be the clone of projections on $\{0, 1\}$. Then the following statements hold.

1. $\mathcal{L}_{(C,\mathrm{J})}$ *is finite if and only if C contains the discriminator function.*
2. $\mathcal{L}_{(C,\mathrm{J})}$ *is countably infinite if and only if $\langle \wedge, \vee \rangle \subseteq C \subseteq \langle \wedge, \vee, 0, 1 \rangle$.*
3. $\mathcal{L}_{(C,\mathrm{J})}$ *has the cardinality of the continuum otherwise.*

Cardinality of the lattice of (C, J)-clonoids



Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	?	?	?	?	?	?	C	F
I	?	?	?	?	?	?	C	F
I^*	?	?	?	?	?	?	C	F
$\Omega(1)$	?	?	?	?	?	?	C	F
V_{01}, Λ_{01}	?	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	?	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	?	?	?	?	?	?	F	F
V, Λ	?	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	?	?	?	?	?	?	F	F
MU^k, MW^k	?	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	?	?	?	?	?	?	F	F
U^k, W^k	?	?	?	?	?	?	F	F
L_{01}	?	?	?	?	?	?	C	F
L_{0*}, L_{*1}	?	?	?	?	?	?	C	F
LS	?	?	?	?	?	?	C	F
L	?	?	?	?	?	?	C	F
SM	?	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	?	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	?	?	?	?	?	?	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	?	?	?	?	?	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Alternation number of a Boolean function

Let $f \colon \{0,1\}^n \to \{0,1\}$.

$P(f, \leq) := (\{0,1\}^n, \leq^n, f)$

The **alternation number** of f, denoted by $\mathrm{Alt}(f)$, is the length of the longest alternating 2-chain in $P(f, \leq)$, where $\leq$ is the chain $0 < 1$.

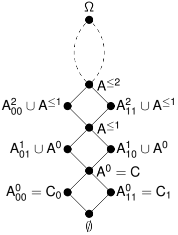
$$\mathsf{A}^k := \{\, f \in \Omega \mid \mathrm{Alt}(f) = k \,\} \qquad \mathsf{A}^{\leq k} := \{\, f \in \Omega \mid \mathrm{Alt}(f) \leq k \,\}$$

(C, J)-clonoids for $C \in \{\mathrm{M}_{01}, \mathrm{M}_{0*}, \mathrm{M}_{*1}, \mathrm{M}\}$

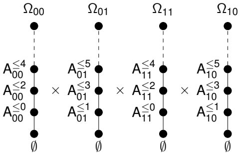
Theorem

1. *The* (M, J)*-clonoids are the sets* $\emptyset$, Ω, $\mathrm{A}^0_{00} = \mathrm{C}_0$, $\mathrm{A}^0_{11} = \mathrm{C}_1$, $\mathrm{A}^{\leq k}$, $\mathrm{A}^{k+1}_{0*} \cup \mathrm{A}^{\leq k}$, *and* $\mathrm{A}^{k+1}_{1*} \cup \mathrm{A}^{\leq k}$ *for* $k \in \mathbb{N}$.
2. *The* $(\mathrm{M}_{0*}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B$*, where*
$$A \in \{\emptyset, \Omega_{0*}\} \cup \{\, \mathrm{A}^{\leq k}_{0*} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{1*}\} \cup \{\, \mathrm{A}^{\leq k}_{1*} \mid k \in \mathbb{N} \,\}.$$
3. *The* $(\mathrm{M}_{*1}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B$*, where*
$$A \in \{\emptyset, \Omega_{*0}\} \cup \{\, \mathrm{A}^{\leq k}_{*0} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{*1}\} \cup \{\, \mathrm{A}^{\leq k}_{*1} \mid k \in \mathbb{N} \,\}.$$
4. *The* $(\mathrm{M}_{01}, \mathrm{J})$*-clonoids are the sets of the form* $A \cup B \cup C \cup D$*, where*
$$A \in \{\emptyset, \Omega_{00}\} \cup \{\, \mathrm{A}^{\leq k}_{00} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{01}\} \cup \{\, \mathrm{A}^{\leq k}_{01} \mid k \in \mathbb{N} \,\},$$
$$C \in \{\emptyset, \Omega_{11}\} \cup \{\, \mathrm{A}^{\leq k}_{11} \mid k \in \mathbb{N} \,\}, \quad D \in \{\emptyset, \Omega_{10}\} \cup \{\, \mathrm{A}^{\leq k}_{10} \mid k \in \mathbb{N} \,\}.$$

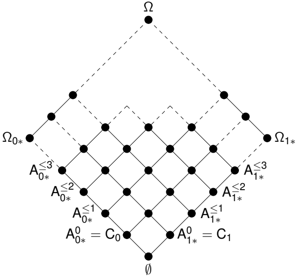
(C, J)-clonoids for $C \in \{\mathrm{M}_{01}, \mathrm{M}_{0*}, \mathrm{M}_{*1}, \mathrm{M}\}$



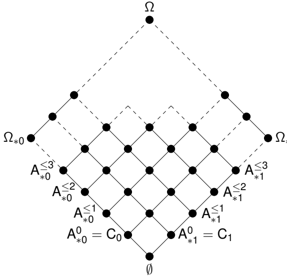
(M, J)-clonoids



$(\mathrm{M}_{01}, \mathrm{J})$-clonoids



$(\mathrm{M}_{0*}, \mathrm{J})$-clonoids



$(\mathrm{M}_{*1}, \mathrm{J})$-clonoids

(C_1, C_2)-clonoids for $\mathsf{M}_{01} \subseteq C_1 \subseteq \mathsf{M}$, $C_2 \subseteq \Omega(1)$

Proposition

1. *For clones* $C_1 \in \{\mathsf{M}, \mathsf{M}_{0*}, \mathsf{M}_{*1}, \mathsf{M}_{01}\}$,
   - *the* (C_1, I_0)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C}_0 = \mathsf{A}^0_{00} \subseteq K$,
   - *the* (C_1, I_1)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C}_1 = \mathsf{A}^0_{11} \subseteq K$,
   - *the* (C_1, I)*-clonoids are* $\emptyset$ *and those* (C_1, J)*-clonoids* K *with* $\mathsf{C} = \mathsf{A}^0 \subseteq K$.
2. *For clones* $C_1 \in \{\mathsf{M}, \mathsf{M}_{0*}, \mathsf{M}_{*1}\}$ *and* $C_2 \in \{\mathsf{I}^*, \Omega(1)\}$, *the* (C_1, C_2)*-clonoids are the sets* $\emptyset$, Ω, $\mathsf{A}^{\leq k}$, *for* $k \in \mathbb{N}$.
3. *The* $(\mathsf{M}_{01}, \mathsf{I}^*)$*-clonoids are the sets of the form* $A \cup B$, *where*
$$A \in \{\emptyset, \Omega_=\} \cup \{\, \mathsf{A}^{\leq k}_{00} \cup \mathsf{A}^{\leq k}_{11} \mid k \in \mathbb{N} \,\},\ B \in \{\emptyset, \Omega_{\neq}\} \cup \{\, \mathsf{A}^{\leq k}_{01} \cup \mathsf{A}^{\leq k}_{10} \mid k \in \mathbb{N} \,\}.$$
4. *The* $(\mathsf{M}_{01}, \Omega(1))$*-clonoids are* $\emptyset$ *and the sets of the form* $A \cup B$, *where*
$$A \in \{\Omega_=\} \cup \{\, \mathsf{A}^{\leq k}_{00} \cup \mathsf{A}^{\leq k}_{11} \mid k \in \mathbb{N} \,\}, \quad B \in \{\emptyset, \Omega_{\neq}\} \cup \{\, \mathsf{A}^{\leq k}_{01} \cup \mathsf{A}^{\leq k}_{10} \mid k \in \mathbb{N} \,\}.$$

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^{\infty}, \mathsf{MU}^{\infty}]$ $[\mathsf{MW}_{01}^{\infty}, \mathsf{MW}^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^{k}, \mathsf{MW}_{01}^{k}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^{k}, MW_{01}^{k}	U	?	?	?	?	?	F	F
MU^{k}, MW^{k}	U	?	?	?	?	?	F	F
U_{01}^{k}, W_{01}^{k}	U	?	?	?	?	?	F	F
U^{k}, W^{k}	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	?	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

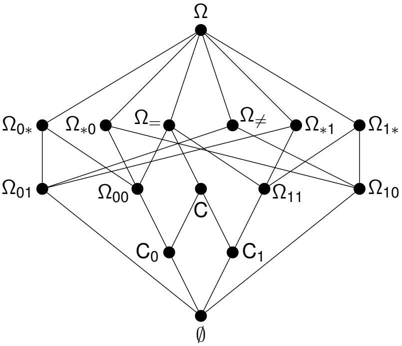
Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^\infty, \mathrm{MU}^\infty]$ $[\mathrm{MW}_{01}^\infty, \mathrm{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^k, \mathrm{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	?	?	?	?	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Example: (M_{01}, L_{01})-clonoids

Proposition

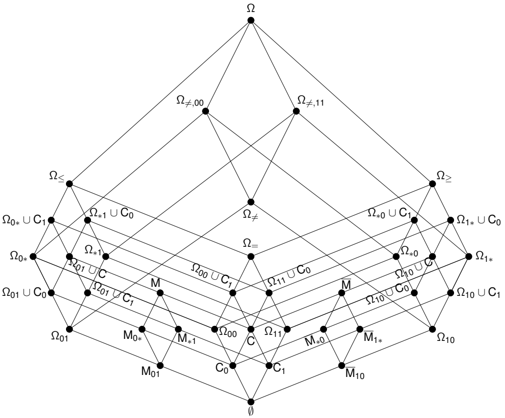
There are precisely 15 (M_{01}, L_{01})*-clonoids, and they are the following:* Ω, Ω_{0*}, Ω_{1*}, Ω_{*0}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, Ω_{00}, Ω_{01}, Ω_{10}, Ω_{11}, C, C_0, C_1, $\emptyset$.



Example: $(\mathrm{M}_{01}, \mathrm{SM})$-clonoids

Proposition

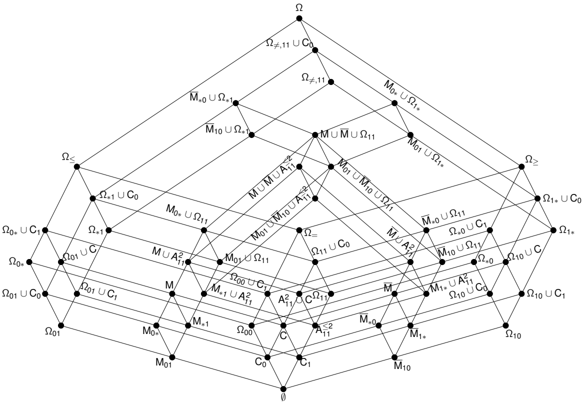
There are precisely 39 $(\mathrm{M}_{01}, \mathrm{SM})$*-clonoids, and they are the following:* Ω, $\Omega_{\leq}$, $\Omega_{\geq}$, $\Omega_{\neq,00}$, $\Omega_{\neq,11}$, $\Omega_{0*} \cup \mathrm{C}_1$, $\Omega_{*0} \cup \mathrm{C}_1$, $\Omega_{1*} \cup \mathrm{C}_0$, $\Omega_{*1} \cup \mathrm{C}_0$, Ω_{0*}, Ω_{*0}, Ω_{1*}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_0$, $\Omega_{10} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}_1$, $\Omega_{00} \cup \mathrm{C}_1$, $\Omega_{11} \cup \mathrm{C}_0$, Ω_{00}, Ω_{11}, Ω_{01}, Ω_{10}, M, M_{0*}, M_{*1}, M_{01}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{*0}$, $\overline{\mathrm{M}}_{1*}$, $\overline{\mathrm{M}}_{10}$, C, C_0, C_1, $\emptyset$.



Example: $(\mathrm{M}_{01}, \mathrm{V}_{01})$-clonoids

Proposition

There are precisely 56 $(\mathrm{M}_{01}, \mathrm{V}_{01})$*-clonoids, and they are the following:* Ω, $\Omega_{\neq,11} \cup \mathrm{C}_0$, $\Omega_{\neq,11}$, $\Omega_{\geq}$, $\Omega_{\leq}$, $\Omega_{=}$, $\Omega_{0*} \cup \mathrm{C}_1$, $\Omega_{*0} \cup \mathrm{C}_1$, $\Omega_{1*} \cup \mathrm{C}_0$, $\Omega_{*1} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}$, $\Omega_{01} \cup \mathrm{C}_0$, $\Omega_{01} \cup \mathrm{C}_1$, $\Omega_{10} \cup \mathrm{C}$, $\Omega_{10} \cup \mathrm{C}_0$, $\Omega_{10} \cup \mathrm{C}_1$, $\Omega_{00} \cup \mathrm{C}_1$, $\Omega_{11} \cup \mathrm{C}_0$, Ω_{0*}, Ω_{*0}, Ω_{1*}, Ω_{*1}, Ω_{01}, Ω_{10}, Ω_{00}, Ω_{11}, $\mathrm{M} \cup \overline{\mathrm{M}} \cup \Omega_{11}$, $\mathrm{M} \cup \overline{\mathrm{M}} \cup \mathrm{A}_{11}^{2}$, $\mathrm{M}_{01} \cup \overline{\mathrm{M}}_{10} \cup \Omega_{11}$, $\mathrm{M}_{01} \cup \overline{\mathrm{M}}_{10} \cup \mathrm{A}_{11}^{\leq 2}$, $\mathrm{M} \cup \mathrm{A}_{11}^{2}$, $\mathrm{M}_{0*} \cup \Omega_{1*}$, $\mathrm{M}_{0*} \cup \Omega_{11}$, $\mathrm{M}_{*1} \cup \mathrm{A}_{11}^{2}$, $\overline{\mathrm{M}} \cup \mathrm{A}_{11}^{2}$, $\overline{\mathrm{M}}_{*0} \cup \Omega_{*1}$, $\overline{\mathrm{M}}_{*0} \cup \Omega_{11}$, $\overline{\mathrm{M}}_{1*} \cup \mathrm{A}_{11}^{2}$, $\mathrm{M}_{01} \cup \Omega_{1*}$, $\mathrm{M}_{01} \cup \Omega_{11}$, $\overline{\mathrm{M}}_{10} \cup \Omega_{*1}$, $\overline{\mathrm{M}}_{10} \cup \Omega_{11}$, M, M_{0*}, M_{*1}, M_{01}, $\overline{\mathrm{M}}$, $\overline{\mathrm{M}}_{*0}$, $\overline{\mathrm{M}}_{1*}$, $\overline{\mathrm{M}}_{10}$, $\mathrm{A}_{11}^{2} \cup \mathrm{C}$, $\mathrm{A}_{11}^{\leq 2}$, C, C_0, C_1, $\emptyset$.



Proposition

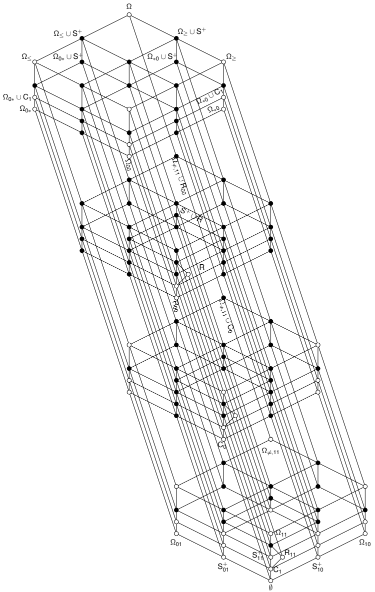
There are precisely 56 $(\mathrm{M}_{01}, \Lambda_{01})$*-clonoids, and they are the duals of the* $(\mathrm{M}_{01}, \mathrm{V}_{01})$*-clonoids.*

Theorem

There are precisely 123 (S_{01}, V_{01})-clonoids. They are precisely the classes

$$\Omega, \quad \Omega_{\leq} \cup S^{+}, \quad \Omega_{\geq} \cup S^{+},$$
$$\Omega_{\leq}, \quad \Omega_{\geq}, \quad \Omega_{0*} \cup S^{+}, \quad \Omega_{*0} \cup S^{+},$$
$$\Omega_{0*} \cup C_1, \quad \Omega_{*0} \cup C_1, \quad \Omega_{0*}, \quad \Omega_{*0},$$
$$\Omega_{\neq,11} \cup R_{00}, \quad \Omega_{\neq,11} \cup C_0, \quad \Omega_{\neq,11},$$
$$S^{+} \cup R, \quad R,$$

and their intersections.



Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	?	?	?	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	?	?	?	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	?	?	?	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	?	?	?	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	?	?	?	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	?	?	?	?	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}_{01}^\infty, \mathsf{MU}^\infty]$ $[\mathsf{MW}_{01}^\infty, \mathsf{MW}^\infty]$	U_{01}^∞ W_{01}^∞	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}_{01}^k, \mathsf{MW}_{01}^k\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	F	F	F	F	F
MU_{01}^k, MW_{01}^k	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U_{01}^k, W_{01}^k	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	?	?	?	?	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	F	F	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Uncountable clonoid lattices

Theorem

For clones C_1 and C_2 such that $C_1 \subseteq K_1$ and $C_2 \subseteq K_2$ for some

$$(K_1, K_2) \in \{(\Omega(1), \Omega(1)), (\Omega(1), \Lambda), (\Omega(1), \mathrm{V}), (\mathrm{I}^*, \mathrm{U}^\infty), (\mathrm{I}^*, \mathrm{W}^\infty), (\mathrm{I}_0, \mathrm{U}^\infty), (\mathrm{I}_1, \mathrm{U}^\infty), (\mathrm{I}_0, \mathrm{W}^\infty), (\mathrm{I}_1, \mathrm{W}^\infty), (\mathrm{L}, \Omega(1)), (\Lambda, \Lambda), (\Lambda, \mathrm{V}), (\mathrm{V}, \Lambda), (\mathrm{V}, \mathrm{V}), (\Lambda, \Omega(1)), (\mathrm{V}, \Omega(1)), (\mathrm{L}, \Lambda), (\mathrm{L}_{0*}, \mathrm{U}^\infty), (\mathrm{L}_{*1}, \mathrm{U}^\infty), (\mathrm{LS}, \mathrm{U}^\infty), (\mathrm{L}, \mathrm{V}), (\mathrm{L}_{0*}, \mathrm{W}^\infty), (\mathrm{L}_{*1}, \mathrm{W}^\infty), (\mathrm{LS}, \mathrm{W}^\infty)\}$$

there are an uncountable infinitude of (C_1, C_2)-clonoids.

E. LEHTONEN, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones. arXiv:2412.01107

E. LEHTONEN, Clonoids of Boolean functions with a linear source clone and a semilattice or 0- or 1-separating target clone. arXiv:2504.04481

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1,C_2)} = \langle T \rangle_{(C_1,C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2)-clonoids.

One of the following families of functions does the job in each case.

- $f_n \colon \{0,1\}^n \to \{0,1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n \colon \{0,1\}^n \to \{0,1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n \colon \{0,1\}^n \to \{0,1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (\mathrm{L}, \Lambda)$.

Uncountable clonoid lattices

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1,C_2)} = \langle T \rangle_{(C_1,C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2)-clonoids.

One of the following families of functions does the job in each case.

- $f_n \colon \{0,1\}^n \to \{0,1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n-1\}$
- $q_n \colon \{0,1\}^n \to \{0,1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n \colon \{0,1\}^n \to \{0,1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (\mathsf{L}, \Lambda)$.

Uncountable clonoid lattices

Proof idea

We exhibit a countably infinite family F of functions with the property that for all $S, T \subseteq F$, $\langle S \rangle_{(C_1, C_2)} = \langle T \rangle_{(C_1, C_2)}$ if and only if $S = T$. Because the power set of a countably infinite set is uncountable, it follows that there is an uncountable infinitude of (C_1, C_2)-clonoids.

One of the following families of functions does the job in each case.

- $f_n \colon \{0, 1\}^n \to \{0, 1\}$, $f_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n - 1\}$
- $q_n \colon \{0, 1\}^n \to \{0, 1\}$, $q_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, n\}$
- $\beta_n \colon \{0, 1\}^n \to \{0, 1\}$, $\beta_n(\mathbf{a}) = 1$ iff $w(\mathbf{a}) \in \{1, 2, n\}$

For the sake of illustration, let us look more carefully into the situation when $(C_1, C_2) = (\mathsf{L}, \Lambda)$.

Uncountable clonoid lattices

Definition

For $n \in \mathbb{N}^+$, define $\beta_n \colon \{0,1\}^n \to \{0,1\}$ by the rule $\beta_n(\mathbf{a}) = 1$ if and only if $w(\mathbf{a}) \in \{1, 2, n\}$.

For $N \subseteq \mathbb{N}$, let $B_N := \{\, \beta_n \mid n \in N \,\}$.

Lemma

Let N be the set of odd integers greater than 6. Let $S \subseteq N$ and $m \in N$. We have $\beta_m \in \langle B_S \rangle_{(\mathsf{L},\Lambda)}$ if and only if $m \in S$.

Proof

Sufficiency.

If $m \in S$, then clearly $\beta_m \in B_S \subseteq \langle B_S \rangle_{(C_1, C_2)}$ for all clones C_1 and C_2.

(*Continued on next slide.*)

Uncountable clonoid lattices

Claim

Let m and n be odd integers greater than 6, and assume that $\beta_m \leqq \gamma = \beta_n \circ \mathbf{g}$, where $\mathbf{g} = (g_1, \dots, g_n) \colon \{0, 1\}^m \to \{0, 1\}^n$ with $g_1, \dots, g_n \in \mathrm{L}$ such that $(\beta_n \circ \mathbf{g})(\mathbf{0}) = 0$. Then $m = n$.

Proof of the Claim. A combinatorial argument with a sprinkle of algebra, too lengthy to present here.

Uncountable clonoid lattices

Proof of the Lemma (continued)

Necessity.

Assume $\beta_m \in \langle B_S \rangle_{(\mathrm{L},\Lambda)} = \Lambda(B_S\,\mathrm{L})$.

Then $\beta_m = \bigwedge_{i=1}^{k} \gamma_i$ for some $k \in \mathbb{N}^+$, where, for $i \in [k]$, $\gamma_i = \beta_{n_i} \circ \mathbf{g}^i$, where $n_i \in S$ and $\mathbf{g}^i = (g^i_1, \dots, g^i_{n_i})$ for some $g^i_1, \dots, g^i_{n_i} \in \mathrm{L}$.

We have $\beta_m \leq \gamma_i$ for all $i \in [k]$, and for every $\mathbf{a} \in \beta_m^{-1}(0)$, there exists an $i \in [k]$ such that $\gamma_i(\mathbf{a}) = 0$.

In particular, there is a $j \in [k]$ such that $\gamma_j(\mathbf{0}) = 0$, that is, $(\beta_{n_j} \circ \mathbf{g}^j)(\mathbf{0}) = 0$.

By our Claim, $m = n_j$, so $m \in S$. □

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}^\infty_{01}, \mathrm{MU}^\infty]$ $[\mathrm{MW}^\infty_{01}, \mathrm{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}^k_{01}, \mathrm{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	?	?	?	?	?	C	F
I	U	?	?	F	F	F	C	F
I^*	U	?	?	?	?	?	C	F
$\Omega(1)$	U	?	?	F	F	F	C	F
V_{01}, Λ_{01}	U	?	?	?	?	?	F	F
V_{0*}, Λ_{*1}	U	?	?	F	F	F	F	F
V_{*1}, Λ_{0*}	U	?	?	?	?	?	F	F
V, Λ	U	?	?	F	F	F	F	F
MU^k_{01}, MW^k_{01}	U	?	?	?	?	?	F	F
MU^k, MW^k	U	?	?	?	?	?	F	F
U^k_{01}, W^k_{01}	U	?	?	?	?	?	F	F
U^k, W^k	U	?	?	?	?	?	F	F
L_{01}	U	?	?	?	?	?	C	F
L_{0*}, L_{*1}	U	?	?	?	?	?	C	F
LS	U	?	?	?	?	?	C	F
L	U	?	?	F	F	F	C	F
SM	U	?	?	?	?	?	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathrm{J}, \mathrm{I}]$	$[\mathrm{I}^*, \Omega(1)]$	$[\mathrm{V}_{01}, \mathrm{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathrm{MU}_{01}^{\infty}, \mathrm{MU}^{\infty}]$ $[\mathrm{MW}_{01}^{\infty}, \mathrm{MW}^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[\mathrm{L}_{01}, \mathrm{L}]$	$[\{\mathrm{SM}, \mathrm{MU}_{01}^{k}, \mathrm{MW}_{01}^{k}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	U	U	U	U	U	C	F
I	U	U	U	F	F	F	C	F
I^*	U	U	U	U	U	U	C	F
$\Omega(1)$	U	U	U	F	F	F	C	F
V_{01}, Λ_{01}	U	U	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	U	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	U	U	U	U	U	F	F
V, Λ	U	U	U	F	F	F	F	F
MU_{01}^{k}, MW_{01}^{k}	U	U	U	U	U	U	F	F
MU^{k}, MW^{k}	U	U	U	U	U	U	F	F
U_{01}^{k}, W_{01}^{k}	U	U	U	U	U	U	F	F
U^{k}, W^{k}	U	U	U	U	U	U	F	F
L_{01}	U	U	U	U	U	U	C	F
L_{0*}, L_{*1}	U	U	U	U	U	U	C	F
LS	U	U	U	U	U	U	C	F
L	U	U	U	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[\mathrm{M}_{01}, \mathrm{M}]$	C	C	F	F	F	F	F	F
$[\mathrm{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Cardinalities of (C_1, C_2)-clonoid lattices

Theorem

Let C_1 and C_2 be clones on $\{0,1\}$. Then the lattice $\mathcal{L}_{(C_1,C_2)}$ of (C_1, C_2)-clonoids is

1. *finite if $C_1 \supseteq K_1$ and $C_2 \supseteq K_2$ for some*

$$(K_1, K_2) \in \{\, (\mathsf{J}, \mathsf{MU}^k_{01}) \mid 2 \leq k < \infty \,\} \cup \{\, (\mathsf{J}, \mathsf{MW}^k_{01}) \mid 2 \leq k < \infty \,\} \cup \{(\mathsf{J}, \mathsf{SM}), (\mathsf{I}, \mathsf{MU}^\infty_{01}), (\mathsf{I}, \mathsf{MW}^\infty_{01}), (\Lambda_{01}, \mathsf{L}_{01}), (\mathsf{V}_{01}, \mathsf{L}_{01}), (\mathsf{V}_{0*}, \mathsf{MU}^\infty_{01}), (\mathsf{V}_{0*}, \mathsf{MW}^\infty_{01}), (\Lambda_{*1}, \mathsf{MU}^\infty_{01}), (\Lambda_{*1}, \mathsf{MW}^\infty_{01}), (\mathsf{M}_{01}, \Lambda_{01}), (\mathsf{M}_{01}, \mathsf{V}_{01}), (\mathsf{S}_{01}, \mathsf{J})\};$$

2. *countably infinite if $C_1 \in [\mathsf{J}, \mathsf{L}]$ and $C_2 \in [\mathsf{L}_{01}, \mathsf{L}]$ or $C_1 \in [\mathsf{M}_{01}, \mathsf{M}]$ and $C_2 \in [\mathsf{J}, \Omega(1)]$;*
3. *uncountable otherwise.*

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{U}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{W}^\infty]$	$[\mathsf{L}_{01}, \mathsf{L}]$	$2 \leq \ell < \infty$ $[\{\mathsf{SM}, \mathsf{MU}^\ell_{01}, \mathsf{MW}^\ell_{01}\}, \Omega]$
$[\mathsf{J}, \{\mathsf{L}_{0*}, \mathsf{L}_{*1}, \mathsf{LS}\}]$	U	U	U	C	F
$[\mathsf{I}, \mathsf{L}]$	U	U	F	C	F
$[\{\Lambda_{01}, \mathsf{V}_{01}, \mathsf{SM}\}, \{\mathsf{U}^2, \mathsf{W}^2\}]$	U	U	U	F	F
$[\Lambda_{*1}, \Lambda]$, $[\mathsf{V}_{0*}, \mathsf{V}]$	U	U	F	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F

Open problems

- Generalize these results for clonoids on arbitrary (finite) base sets. (See the work by S. Kreinecker, S. Fioravanti, P. Mayr, P. Wynne.)
- Find necessary and sufficient conditions for the finiteness of the lattice of (C_1, C_2)-clonoids.
  (The “uniform generation by minors” by S. Fioravanti, M. Kompatscher, B. Rossi gives a sufficient condition.)
- Find simple relational descriptions for clonoids.

Literature

M. Couceiro, E. Lehtonen, Stability of Boolean function classes with respect to clones of linear functions, *Order* **41** (2024) 15–64.

E. Lehtonen, Majority-closed minions of Boolean functions, *Algebra Universalis* **85** (2024) Art. 6.

E. Lehtonen, Near-unanimity-closed minions of Boolean functions, *Algebra Universalis* **86** (2025) Art. 2.

E. Lehtonen, Clonoids of Boolean functions with a monotone or discriminator source clone. *Algebra Universalis* **87** (2026) Art. 4.

E. Lehtonen, Clonoids of Boolean functions with essentially unary, linear, semilattice, or 0- or 1-separating source and target clones. *Internat. J. Algebra Comput.* **36**(1) (2026) 17–50.

E. Lehtonen, Clonoids of Boolean functions with a linear source clone and a semilattice or 0- or 1-separating target clone. *Algebra Universalis* **87** (2026) Art. 15.

The End

Thank you for your attention!